

AN ALGEBRAIC PROOF OF THE HYPERPLANE PROPERTY OF THE GENUS ONE GW-INVARIANTS OF QUINTICS

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ABSTRACT. Li-Zinger's *hyperplane theorem* states that the genus one GW-invariants of the quintic threefold is the sum of its reduced genus one GW-invariants and $1/12$ multiplies of its genus zero GW-invariants. We apply the Guffin-Sharpe-Witten's theory (GSW theory) to give an algebro-geometric proof of the *hyperplane theorem*, including separation of contributions and computation of $1/12$.

1. INTRODUCTION

GW-invariants of a smooth projective variety X are virtual enumeration of stable maps to X . Let $d \in H_2(X, \mathbb{Z})$, and let $\overline{M}_g(X, d)$ be the moduli of genus g stable morphisms to X ; it is a proper, separated DM-stack, and carries a canonical virtual cycle $[\overline{M}_g(X, d)]^{\text{vir}}$. When X is a Calabi-Yau threefold, this class is a dimension zero class, and the degree d genus g GW-invariants of X are

$$N_g(d)_X = \deg[\overline{M}_g(X, d)]^{\text{vir}}.$$

Investigating GW-invariants of Calabi-Yau threefolds is one of the main focus in the subject of Mirror Symmetry. The subclass of genus zero invariants is largely known by now, thanks to the works of [Ko, Gi, LLY], in case the Calabi-Yau threefold X is a complete intersection in a product of projective spaces \mathbb{P} , which can be realized as an equivariant integral of the top Chern class of a vector bundle $\mathbb{E}_{0,d}$ on the moduli of stable morphisms to \mathbb{P} :

$$(1.1) \quad [\overline{M}_0(X, d)]^{\text{vir}} = \int_{[\overline{M}_0(\mathbb{P}, d)]} c_{\text{top}}(\mathbb{E}_{0,d}).$$

We call this property the “*hyperplane property of the GW-invariants*”. The techniques developed in [Gi, LLY] allow one to completely solve the genus zero invariants for such X .

Generalizing this to high genus requires new approach, in part because the *hyperplane property* stated fails for positive genus invariants. In [LZ] Zinger and the second named author introduced the reduced $g = 1$ GW-invariants $N_1(d)_Q^{\text{red}}$ of the quintic $Q \subset \mathbb{P}^4$ by applying the hyperplane property to the primary component of $\overline{M}_1(\mathbb{P}^4, d)$, and proved that the reduced $g = 1$ GW-invariants $N_1(d)_Q^{\text{red}}$ relates to the ordinary GW-invariants $N_1(d)_Q$ by a simple linear relation.

In this paper, we give an algebraic proof of this theorem.

Theorem 1.1. (Li-Zinger) The reduced and the ordinary GW-invariants of a quintic Calabi-Yau $Q \subset \mathbb{P}^4$ are related by

$$N_1(d)_Q = N_1(d)_Q^{\text{red}} + \frac{1}{12} N_0(d)_Q.$$

In algebraic geometry, the reduced genus one GW-invariants of quintics take the following form. Let $f : \mathcal{C} \rightarrow \mathbb{P}^4$ and $\pi : \mathcal{C} \rightarrow \overline{M}_1(\mathbb{P}^4, d)$ be the universal family of $\overline{M}_1(\mathbb{P}^4, d)$; let $\overline{M}_1(\mathbb{P}^4, d)_{\text{pri}} \subset \overline{M}_1(\mathbb{P}^4, d)$ be the primary component that is the closure of all stable morphisms with smooth domains. We then pick a DM-stack $\tilde{\mathcal{X}}_{\text{pri}}$ and a proper birational morphism $\varphi : \tilde{\mathcal{X}}_{\text{pri}} \rightarrow \overline{M}_1(\mathbb{P}^4, d)_{\text{pri}}$ so that with $\tilde{f}_{\text{pri}} : \tilde{\mathcal{C}}_{\text{pri}} \rightarrow \mathbb{P}^4$ and $\tilde{\pi}_{\text{pri}} : \tilde{\mathcal{C}}_{\text{pri}} \rightarrow \tilde{\mathcal{X}}_{\text{pri}}$ the pull back of (f, π) , the direct image sheaf

$$\tilde{\pi}_{\text{pri}*} \tilde{f}_{\text{pri}}^* \mathcal{O}_{\mathbb{P}^4}(5)$$

is locally free on $\tilde{\mathcal{X}}_{\text{pri}}$. According to [VZ], (see also [HL1],) such φ exists. We state the working definition of the reduced invariants of Q .

Definition 1.2. *We define the reduced $g = 1$ GW-invariants of Q be*

$$(1.2) \quad N_1(d)_Q^{\text{red}} = \int_{[\tilde{\mathcal{X}}_{\text{pri}}]} c_{\text{top}}(\tilde{\pi}_{\text{pri}*} \tilde{f}_{\text{pri}}^* \mathcal{O}_{\mathbb{P}^4}(5)).$$

To prove Theorem 1.1, one needs to separate $[\overline{M}_g(Q, d)]^{\text{vir}}$ into its “primary” part and its “ghost” part. One then shows that the “primary” part can be evaluated using (1.2), and shows that the “ghost” part contributes $\frac{1}{12}$ multiple of the genus zero GW-invariants.

The original proof of this theorem uses analytic method, which succeeds the desired separation by perturbing the complex structure of Q to a generic almost complex structure, through hardcore analysis [Zi2].

The proof worked out in this paper uses the Guffin-Sharpet-Witten’s invariants. The GSW invariants originates from Guffin and Sharpe’s pioneer work in [GS]. Later an algebraic geometric treatment is given and GSW invariants can be realized as the GW-invariants of stable maps with p -field developed in [CL1]. Since the structure of the moduli of stable maps to \mathbb{P}^4 with fields is much easier to analyze, it allows us to separate the “primary” and the “ghost” contributions algebraically. We are confident that this method can be pushed to higher genus GW-invariants of quintics and other complete intersections of product of projective spaces.

We now outline our proof. Given a positive integer d , we form the moduli $\overline{M}_1(\mathbb{P}^4, d)^p$ of genus 1 degree d stable morphisms to \mathbb{P}^4 with p -fields:

$$\overline{M}_1(\mathbb{P}^4, d)^p = \{[u, C, p] \mid [u, C] \in \overline{M}_1(\mathbb{P}^4, d), p \in \Gamma(u^* \mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_C)\} / \sim.$$

It is a Deligne-Mumford stack, with a perfect obstruction theory. The polynomial $\mathbf{w} = x_1^5 + \dots + x_5^5$ (or any generic quintic polynomial) induces a cosection (homomorphism) of its obstruction sheaf

$$(1.3) \quad \sigma : \mathcal{O}b_{\overline{M}_1(\mathbb{P}^4, d)^p} \longrightarrow \mathcal{O}_{\overline{M}_1(\mathbb{P}^4, d)^p},$$

whose non-surjective locus (called the degeneracy locus) is

$$\overline{M}_1(Q, d) \subset \overline{M}_1(\mathbb{P}^4, d)^p, \quad Q = (x_1^5 + \dots + x_5^5 = 0) \subset \mathbb{P}^4,$$

which is proper. The cosection localized virtual class construction of Kiem-Li defines a localized virtual cycle

$$[\overline{M}_1(\mathbb{P}^4, d)^p]_{\sigma}^{\text{vir}} \in A_0 \overline{M}_1(Q, d).$$

(For convention of cosection localized virtual cycles, see discussion after (2.6).) The GW-invariant of $\overline{M}_1(\mathbb{P}^4, d)^p$ is

$$N_1(d)_{\mathbb{P}^4}^p = \deg[\overline{M}_1(\mathbb{P}^4, d)^p]_{\sigma}^{\text{vir}}.$$

Theorem 1.3 ([CL1]). *For $d > 0$, the GW-invariant of $\overline{M}_1(\mathbb{P}^4, d)^p$ coincides with the GW-invariant $N_1(d)_Q$ of the quintic Q up to a sign:*

$$N_1(d)_{\mathbb{P}^4}^p = (-1)^{5d} \cdot N_1(d)_Q.$$

By this theorem, to prove Theorem 1.1 it suffices to study the separation of the cycle $[\overline{M}_1(\mathbb{P}^4, d)^p]_{\sigma}^{\text{vir}}$. Following [VZ] and [HL1], we perform a modular blowing-up $\tilde{\mathcal{Y}}$ of $\overline{M}_1(\mathbb{P}^4, d)^p$. The blown-up $\tilde{\mathcal{Y}}$ is a union of $\tilde{\mathcal{Y}}_{\text{pri}}$, which is birational to the primary component of $\overline{M}_1(\mathbb{P}^4, d)$, and other smooth components $\tilde{\mathcal{Y}}_{\mu}$, indexed by partitions μ of d :

$$\tilde{\mathcal{Y}} = \tilde{\mathcal{Y}}_{\text{gst}} \cup (\cup_{\mu \vdash d} \tilde{\mathcal{Y}}_{\mu}).$$

Geometrically, general elements of $\tilde{\mathcal{Y}}_{\text{pri}}$ are stable morphisms in $\overline{M}_1(\mathbb{P}^4, d)$ having smooth domains; general elements of $\tilde{\mathcal{Y}}_{\mu}$ (with $\mu = (d_1, \dots, d_{\ell})$) consist of

- (1). stable morphisms $[u, C] \in \overline{M}_1(\mathbb{P}^4, d)$ whose domains C are smooth elliptic curves attached on with ℓ rational curves, and the morphisms u are constant along the elliptic curves and have degree d_i along the i -th rational curves;
- (2). a p -field $\psi \in \Gamma(C, u^*\mathcal{O}(-5) \otimes \omega_C)$, and
- (3). auxiliary data from the modular blowing-up.

Let \mathcal{D} be the Artin stack of pairs (C, L) of degree d line bundles L on arithmetic genus one nodal curves C . We perform a parallel modular blowing-up of \mathcal{D} to obtain $\tilde{\mathcal{D}} \rightarrow \mathcal{D}$. The pullback of $\mathcal{O}(1)$ via the universal morphism on $\tilde{\mathcal{Y}}$ induces a morphism $\tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{D}}$. By working out the relative perfect obstruction theory of $\tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{D}}$, we obtain its obstruction complex $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$. The intrinsic normal cone $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ of $\tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{D}}$ is embedded in $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})$.

By picking a homogeneous quintic polynomial, say $x_1^5 + \dots + x_5^5$, we construct a cosection (homomorphism) $\tilde{\sigma} : h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}) \rightarrow \mathcal{O}_{\tilde{\mathcal{Y}}}$. By a cosection localized version of [Cos, Thm 5.0.1] we have

$$(1.4) \quad \deg[\overline{M}_1(\mathbb{P}^4, d)^p]_{\sigma}^{\text{vir}} = \deg 0_{\tilde{\sigma}, \text{loc}}^1[\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}].$$

Using the explicit local defining equation of $\tilde{\mathcal{Y}}$, which is obtained following the work of [Zi1, ?], we conclude that the cone $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ is separated into union

$$[\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}] = [\mathbf{C}_{\text{pri}}] + \sum_{\mu \vdash d} [\mathbf{C}_{\mu}],$$

where \mathbf{C}_{pri} is an irreducible cycle lies over $\tilde{\mathcal{Y}}_{\text{gst}}$; each \mathbf{C}_{μ} lies over $\tilde{\mathcal{Y}}_{\mu}$. Thus

$$(1.5) \quad \deg 0_{\tilde{\sigma}, \text{loc}}^1[\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}] = 0_{\tilde{\sigma}, \text{loc}}^1[\mathbf{C}_{\text{pri}}] + \sum_{\mu \vdash d} 0_{\tilde{\sigma}, \text{loc}}^1[\mathbf{C}_{\mu}].$$

In Section four, we show that $0_{\tilde{\sigma}, \text{loc}}^1[\mathbf{C}_{\text{pri}}]$ equals the reduced GW-invariant defined in (1.4); in Section six, we show that $0_{\tilde{\sigma}, \text{loc}}^1[\mathbf{C}_{\mu}] = 0$ for all partitions $\mu \neq (d)$, where (d) is the partition of d into a single block, and finally in Section seven, we prove that $0_{\tilde{\sigma}, \text{loc}}^1[\mathbf{C}_{(d)}] = \frac{(-1)^{5d}}{12} N_0(d)_Q$. Together with (1.4), (1.5) and Theorem 1.3, we prove the main Theorem 1.1.

The authors' original (algebraic-geometric) approach to prove Theorem 1.1 is to introduce an auxiliary substack of $\overline{M}_1(\mathbb{P}^4, d)$ to capture the contribution $\frac{1}{12} N_1(d)_Q$. Using this, the authors showed that the union of this auxiliary substack with the

primary component of $\overline{M}_1(\mathbb{P}^4, d)$ contains $\overline{M}_1(Q, d)$, and has (in principle) a perfect relative obstruction theory, thus it has the “hyperplane property” induced by the relative obstruction theory. However, the theories developed to study the deformation theory of the union of this auxiliary substack with the primary component of $\overline{M}_1(\mathbb{P}^4, d)$ were far from satisfactory, and were never finalized.

After the discovery of the GW-theory of stable maps with p -field (Guffin-Sharpe-Witten invariants) and its equivalence with the GW-invariants of quintics, we use the new moduli spaces and their localized virtual cycles to provide the current proof of Theorem 1.1. We expect that this approach can be generalized to prove a conjecture of Zinger and the second named author on high genus reduced GW-invariants of quintics and other complete intersection Calabi-Yau threefolds in the product of projective spaces. For instance, the partial blowing-up of moduli of genus two stable maps to \mathbb{P}^n [HL2] should lead to a proof of this conjecture for the genus two case.

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2. MODULI OF STABLE MORPHISMS WITH FIELDS

We recall the moduli of stable morphisms with fields and its associated invariants introduced in [CL1]. To simplify the notation, we will focus on the genus one case.

We begin with the moduli of stable maps. Let $\overline{M}_1(\mathbb{P}^4, d)$ be the moduli of genus one degree d stable maps to \mathbb{P}^4 . In this paper, we abbreviate it to $\mathcal{X} := \overline{M}_1(\mathbb{P}^4, d)$, with $g = 1$ and d implicitly understood. We denote by

$$(f_{\mathcal{X}}, \pi_{\mathcal{X}}) : \mathcal{C}_{\mathcal{X}} \rightarrow \mathbb{P}^4 \times \mathcal{X}$$

its universal family of \mathcal{X} .

We recall the definition of the Moduli of stable morphisms with fields.

Definition-Proposition 2.1. *We let $\overline{M}_1(\mathbb{P}^4, d)^p$ be the groupoid that associates to any scheme S the set $\overline{M}_1(\mathbb{P}^4, d)^p(S)$ of all S -families $(\mathcal{C}_S, f_S, \psi_S)$ where (1). $[\mathcal{C}_S, f_S] \in \overline{M}_1(\mathbb{P}^4, d)(S)$, and (2). $\psi_S \in \Gamma(\mathcal{C}_S, f_S^* \mathcal{O}(-5) \otimes \omega_{\mathcal{C}_S/S})$. An arrow from $(\mathcal{C}_S, f_S, \psi_S)$ to $(\mathcal{C}'_S, f'_S, \psi'_S)$ consists of an arrow η from (\mathcal{C}_S, f_S) to (\mathcal{C}'_S, f'_S) such that $\eta^*(\psi'_S) = \psi_S$.*

The groupoid $\overline{M}_1(\mathbb{P}^4, d)^p$ is a separated Deligne-Mumford stack of finite type.

In [CL1], this moduli space is constructed as the direct image cone over the the Artin stack of pairs of curves with line bundles. Let \mathcal{D} be the groupoid whose objects $\tau \in \mathcal{D}(S)$, S is a scheme, are pairs $(\mathcal{C}_{\tau}, \mathcal{L}_{\tau})$, where $\pi_{\tau} : \mathcal{C}_{\tau} \rightarrow S$ is a flat family of genus one connected nodal curves and \mathcal{L}_{τ} a fiberwise degree d line bundle on \mathcal{C}_{τ} ; an arrow from τ to τ' in $\mathcal{D}(S)$ consists of an S -isomorphism $\phi_1 : \mathcal{C}_{\tau} \rightarrow \mathcal{C}_{\tau'}$ and an isomorphism $\phi_2 : \mathcal{L}_{\tau} \rightarrow \phi_1^* \mathcal{L}_{\tau'}$. Note that the invertible sheaf $\mathcal{L}_{\mathcal{X}} := f_{\mathcal{X}}^* \mathcal{O}(1)$ on $\mathcal{C}_{\mathcal{X}}$ induces a tautological morphism $\mathcal{X} \rightarrow \mathcal{D}$.

We let $\pi : \mathcal{C} \rightarrow \mathcal{D}$ with \mathcal{L} on \mathcal{C} be the universal curve and line bundle of \mathcal{D} ; we introduce an auxiliary invertible sheaf $\mathcal{P} = \mathcal{L}^{\otimes(-5)} \otimes \omega_{\mathcal{C}/\mathcal{D}}$. We define the direct

image stack

$$(2.1) \quad C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P}))$$

to be the category whose objects in $C(\pi_*(\mathcal{L}^{\oplus 4} \oplus \mathcal{P}))(S)$ consists $(\mathcal{C}_\tau, \mathcal{L}_\tau, u_i, \psi)$, where $\tau \in \mathcal{D}(S)$ with $\pi_\tau : \mathcal{C}_\tau \rightarrow S$ and \mathcal{L}_τ on \mathcal{C}_τ its associated families, $(u_1, \dots, u_5) \in \Gamma(\pi_{\tau*}\mathcal{L}_\tau^{\oplus 5})$ and $\psi \in \Gamma(\pi_{\tau*}\mathcal{P}_\tau)$. An arrow from $(\mathcal{C}_\tau, \mathcal{L}_\tau, u_i, \psi)$ to $(\mathcal{C}_{\tau'}, \mathcal{L}_{\tau'}, u'_i, \psi')$ (over the same S) consists of an arrow $\tau \rightarrow \tau'$ in $\mathcal{D}(S)$ plus the identity of (u_i, ψ) with the pullback of (u'_i, ψ') under the mentioned arrow $\tau \rightarrow \tau'$. By construction, $C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P}))$ is a stack over \mathcal{D} .

For simplicity, in this paper we abbreviate $\mathcal{Y} = \overline{M}_1(\mathbb{P}^4, d)^p$, with $g = 1$ and d implicitly understood. Like before, we let

$$(f_{\mathcal{Y}}, \pi_{\mathcal{Y}}) : \mathcal{C}_{\mathcal{Y}} \longrightarrow \mathbb{P}^4 \times \mathcal{Y}, \quad \psi_{\mathcal{Y}} \in \Gamma(\mathcal{C}_{\mathcal{Y}}, \mathcal{P}_{\mathcal{Y}}) \quad \text{and} \quad \mathcal{P}_{\mathcal{Y}} = f_{\mathcal{Y}}^* \mathcal{O}(-5) \otimes \omega_{\mathcal{C}_{\mathcal{Y}}/\mathcal{Y}}$$

be the universal family of \mathcal{Y} . We denote $\mathcal{L}_{\mathcal{Y}} = f_{\mathcal{Y}}^* \mathcal{O}(1)$. After fixing a homogeneous coordinates $[z_1, \dots, z_5]$ of \mathbb{P}^4 , the morphism $f_{\mathcal{Y}}$ is given by $(u_{\mathcal{Y}, i}, u_{\mathcal{Y}, i} = f_{\mathcal{Y}}^* z_i)$. The data $(\mathcal{C}_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}}, u_{\mathcal{Y}, i}, \psi_{\mathcal{Y}})$ induces an open embedding $\overline{M}_1(\mathbb{P}^4, d)^p \rightarrow C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P}))$. We let p be the composite

$$p : \mathcal{Y} = \overline{M}_1(\mathbb{P}^4, d)^p \subset C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P})) \longrightarrow \mathcal{D}.$$

After working out the obstruction theory of $C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P}))$ relative to \mathcal{D} (cf. [CL1, Prop 3.1]), we obtain a perfect relative obstruction theory (of $\mathcal{Y} \rightarrow \mathcal{D}$)

$$(2.2) \quad \phi_{\mathcal{Y}/\mathcal{D}} : (E_{\mathcal{Y}/\mathcal{D}})^{\vee} \longrightarrow L_{\mathcal{Y}/\mathcal{D}}^{\bullet}, \quad E_{\mathcal{Y}/\mathcal{D}} := R^{\bullet} \pi_{\mathcal{Y}*}(\mathcal{L}_{\mathcal{Y}}^{\oplus 5} \oplus \mathcal{P}_{\mathcal{Y}}).$$

In the same spirit, a relative perfect obstruction theory of $\mathcal{X} \rightarrow \mathcal{D}$ (cf. [CL1, Prop 2.5 and 2.7]) is (denoting $\mathcal{L}_{\mathcal{X}} = f_{\mathcal{X}}^* \mathcal{O}(1)$):

$$(2.3) \quad \phi_{\mathcal{X}/\mathcal{D}} : (E_{\mathcal{X}/\mathcal{D}})^{\vee} \longrightarrow L_{\mathcal{X}/\mathcal{D}}^{\bullet}, \quad E_{\mathcal{X}/\mathcal{D}} := R^{\bullet} \pi_{\mathcal{X}*} \mathcal{L}_{\mathcal{X}}^{\oplus 5}.$$

According to the convention, we call the cohomology sheaf

$$\mathcal{O}b_{\mathcal{Y}/\mathcal{D}} := H^1(E_{\mathcal{Y}/\mathcal{D}}) = R^1 \pi_{\mathcal{Y}*}(\mathcal{L}_{\mathcal{Y}}^{\oplus 5} \oplus \mathcal{P}_{\mathcal{Y}})$$

the relative obstruction sheaf of $\phi_{\mathcal{Y}/\mathcal{D}}$.

Since \mathcal{Y} is non-proper, we use Kiem and the second named author's cosection localized virtual class to construct its GW-invariants. In [CL1], the authors have constructed a cosection of $\mathcal{O}b_{\mathcal{Y}/\mathcal{D}}$; namely, a homomorphism

$$(2.4) \quad \sigma : \mathcal{O}b_{\mathcal{Y}/\mathcal{D}} \longrightarrow \mathcal{O}_{\mathcal{Y}},$$

based on a choice of a quintic polynomial, say $\mathbf{w} = x_1^5 + \dots + x_5^5$. It was verified in the same paper that this cosection lifted to a cosection $\bar{\sigma} : \mathcal{O}b_{\mathcal{Y}} \rightarrow \mathcal{O}_{\mathcal{Y}}$ of the (absolute) obstruction sheaf $\mathcal{O}b_{\mathcal{Y}}$, where $\mathcal{O}b_{\mathcal{Y}}$ is defined by the exact sequence

$$p^* \Omega_{\mathcal{D}}^{\vee} \longrightarrow \mathcal{O}b_{\mathcal{Y}/\mathcal{D}} \longrightarrow \mathcal{O}b_{\mathcal{Y}} \longrightarrow 0.$$

It was also verified that the degeneracy locus $D(\sigma)$ of σ , which is the locus where σ is not surjective, is the closed subset

$$(2.5) \quad D(\sigma) = \overline{M}_1(Q, d) \subset \overline{M}_1(\mathbb{P}^4, d)^p = \mathcal{Y}.$$

Here $Q = (\mathbf{w} = 0) \subset \mathbb{P}^4$ is the smooth Calabi-Yau quintic defined by the vanishing of the quintic polynomial $\mathbf{w} = x_1^5 + \dots + x_5^5$ used to construct the cosection σ ; $\overline{M}_1(Q, d)$ is the moduli of stable morphisms to Q , and the embedding is via the tautological embedding $\overline{M}_1(Q, d) \subset \overline{M}_1(\mathbb{P}^4, d)$ composed with the embedding $\overline{M}_1(\mathbb{P}^4, d) \subset \mathcal{Y} = \overline{M}_1(\mathbb{P}^4, d)^p$ defined by assigning $\psi = 0$.

The cosection σ induces a morphism of bundle-stack (see [BF] for bundle stacks)

$$(2.6) \quad \sigma : h^1/h^0(E_{\mathcal{Y}/\mathcal{D}}) \longrightarrow \mathcal{O}_{\mathcal{Y}}$$

that is surjective over $\mathcal{U} = \mathcal{Y} - D(\sigma)$. (By abuse of notation, we use the same σ ; also, we use $\mathcal{O}_{\mathcal{Y}}$ to denote the rank one trivial line bundle on \mathcal{Y} .) We let

$$h^1/h^0(E_{\mathcal{Y}/\mathcal{D}})_{\sigma} = (h^1/h^0(E_{\mathcal{Y}/\mathcal{D}}) \times_{\mathcal{Y}} D(\sigma)) \cup \ker\{\sigma|_{\mathcal{U}} : h^1/h^0(E_{\mathcal{Y}/\mathcal{D}})|_{\mathcal{U}} \rightarrow \mathcal{O}_{\mathcal{U}}\},$$

endowed with reduced stack structure.

Applying [KL] on cosection localized virtual class, we know that the intrinsic normal cone (cycle) $[\mathbf{C}_{\mathcal{Y}/\mathcal{D}}] \in Z_* h^1/h^0(E_{\mathcal{Y}/\mathcal{D}})_{\sigma}$ lies in

$$(2.7) \quad [\mathbf{C}_{\mathcal{Y}/\mathcal{D}}] \in Z_* h^1/h^0(E_{\mathcal{Y}/\mathcal{D}})_{\sigma};$$

applying the localized Gysin map

$$0_{\sigma, \text{loc}}^! : A_* h^1/h^0(E_{\mathcal{Y}/\mathcal{D}})_{\sigma} \longrightarrow A_{*-n} D(\sigma),$$

where $-n = \text{rank } E_{\mathcal{Y}/\mathcal{D}}$, we obtain a localized virtual class

$$(2.8) \quad [\mathcal{Y}]_{\text{loc}}^{\text{vir}} := 0_{\sigma, \text{loc}}^! [\mathbf{C}_{\mathcal{Y}/\mathcal{D}}] \in A_0 D(\sigma) = A_0 \overline{M}_1(Q, d).$$

We define its degree to be the GW-invariants of stable morphisms to \mathbb{P}^4 with fields:

$$N_1(d)_{\mathbb{P}^4}^p = \deg [\mathcal{Y}]_{\text{loc}}^{\text{vir}}.$$

This is well-defined since $\overline{M}_1(Q, d)$ is proper.

Theorem 2.2 ([CL1]). *Let $N_1(d)_Q$ be the GW-invariants of genus one degree d stable morphisms to Q , then we have*

$$N_1(d)_{\mathbb{P}^4}^p = (-1)^{5d} N_1(d)_Q.$$

We remark that this Theorem holds for all genus g . For our purpose, we only state in the case $g = 1$.

We will use the modular blow-up of $\mathcal{X} = \overline{M}(\mathbb{P}^4, d)$ to study $N_1(d)_{\mathbb{P}^4}^p$. The version we use is that worked out by Hu and the second named author [HL1], following the original construction of Vakil-Zinger's modular blow-up of the primary component of \mathcal{X} [VZ].

Let \mathcal{M}^w be the Artin stack of weighted genus one curves. (A weighted nodal curve is a nodal curve with its irreducible components assigned non-negative weights; the sum of the weights of all irreducible components is called the total weight.) Replacing \mathcal{L} (of the universal family $(\mathcal{C}, \mathcal{L})$ of \mathcal{D}) by its degrees along irreducible components of fibers of $\mathcal{C} \rightarrow \mathcal{D}$, we obtain an induced morphism $\mathcal{D} \rightarrow \mathcal{M}^w$. Let

$$(2.9) \quad \mathcal{X} \longrightarrow \mathcal{M}^w, \quad \mathcal{Y} \longrightarrow \mathcal{M}^w$$

be the composites of $\mathcal{X}, \mathcal{Y} \rightarrow \mathcal{D}$ with $\mathcal{D} \rightarrow \mathcal{M}^w$.

Let $\tilde{\mathcal{M}}^w$ be modular blow up of \mathcal{M}^w described in [HL1]. (Since we do not need the explicit form of this blow-up in this paper, we will be content to describe the consequence of this blow-up.) We define the modular blow-up of \mathcal{X} and \mathcal{Y} to be

$$\tilde{\mathcal{X}} = \mathcal{X} \times_{\mathcal{M}^w} \tilde{\mathcal{M}}^w, \quad \tilde{\mathcal{Y}} = \mathcal{Y} \times_{\mathcal{M}^w} \tilde{\mathcal{M}}^w, \quad \text{and} \quad \tilde{\mathcal{D}} = \mathcal{D} \times_{\mathcal{M}^w} \tilde{\mathcal{M}}^w.$$

We denote $\mathcal{C}_{\tilde{\mathcal{Y}}} = \mathcal{C}_{\mathcal{Y}} \times_{\mathcal{Y}} \tilde{\mathcal{Y}}$ and let $f_{\tilde{\mathcal{Y}}} : \mathcal{C}_{\tilde{\mathcal{Y}}} \rightarrow \mathbb{P}^4$ be the composition of $f_{\mathcal{Y}}$ with $\mathcal{C}_{\tilde{\mathcal{Y}}} \rightarrow \mathcal{C}_{\mathcal{Y}}$. We call $(f_{\tilde{\mathcal{Y}}}, \pi_{\tilde{\mathcal{Y}}}) : \mathcal{C}_{\tilde{\mathcal{Y}}} \rightarrow \mathbb{P}^4 \times \tilde{\mathcal{Y}}$ the tautological family of $\tilde{\mathcal{Y}}$.

Let $\zeta : \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ be the projection. Since $\tilde{\mathcal{Y}}$ is derived from \mathcal{Y} by a base change, we endow the relative obstruction of $\tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{D}}$ the pullback of that of $\mathcal{Y} \rightarrow \mathcal{D}$:

$$(2.10) \quad \phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} : (E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})^\vee \longrightarrow L_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}^\bullet, \quad E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} = \zeta^* E_{\mathcal{Y}/\mathcal{D}}.$$

Thus, $\mathcal{O}b_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} = \zeta^* \mathcal{O}b_{\mathcal{Y}/\mathcal{D}}$; the cosection σ pullbacks to a cosection

$$(2.11) \quad \tilde{\sigma} = \zeta^* \sigma : \mathcal{O}b_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} \rightarrow \mathcal{O}_{\tilde{\mathcal{Y}}},$$

and the degeneracy locus

$$D(\tilde{\sigma}) = D(\sigma) \times_{\mathcal{Y}} \tilde{\mathcal{Y}} = \overline{M}_1(Q, d) \times_{\mathcal{M}^w} \tilde{\mathcal{M}}^w,$$

which is proper. We define $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})_{\tilde{\sigma}}$ parallel to that defined after (2.6). Then by (2.7), we have

$$[\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}] \in Z_* h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})_{\tilde{\sigma}}.$$

We define $[\tilde{\mathcal{Y}}]_{\text{loc}}^{\text{vir}} = 0_{\tilde{\sigma}, \text{loc}}^! [\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}] \in A_0 D(\tilde{\sigma})$ to be the cosection localized virtual class.

Proposition 2.3. *We have identity*

$$\deg[\tilde{\mathcal{Y}}]_{\text{loc}}^{\text{vir}} = \deg[\mathcal{Y}]_{\text{loc}}^{\text{vir}} = (-1)^{5d} N_1(d)_Q.$$

Proof. We offer two proofs. The first is to use Theorem 2.2 and to prove $\deg[\tilde{\mathcal{Y}}]_{\text{loc}}^{\text{vir}} = \deg[\mathcal{Y}]_{\text{loc}}^{\text{vir}}$; the later is proved using the cosection localization version of [Cos, Thm 5.0.1], and that $\tilde{\mathcal{D}} \rightarrow \mathcal{D}$ is a birational morphism between two smooth Artin stacks.

The second proof is to show that $\deg[\tilde{\mathcal{Y}}]_{\text{loc}}^{\text{vir}} = \deg[\tilde{\mathcal{X}}]_{\text{loc}}^{\text{vir}}$, which can be proved parallel to the proof of $\deg[\mathcal{Y}]_{\text{loc}}^{\text{vir}} = \deg[\mathcal{X}]_{\text{loc}}^{\text{vir}}$ worked out in [CL1], and with $\deg[\tilde{\mathcal{X}}]_{\text{loc}}^{\text{vir}} = \deg[\mathcal{X}]_{\text{loc}}^{\text{vir}}$ using [Cos, Thm 5.0.1].

Since both proofs are repetition of the known proofs with routine modifications, we skip the details. \square

3. THE DECOMPOSITION OF CONES

Following [HL1], we know that the blown-up $\tilde{\mathcal{X}}$ is a union of smooth Deligne-Mumford stacks: one is the proper transform of $\mathcal{X}_{\text{pri}} \subset \mathcal{X}$, which we denote by $\tilde{\mathcal{X}}_{\text{pri}}$; the others are indexed by partitions μ of d , denoted by $\tilde{\mathcal{X}}_\mu$. Geometrically, a generic element of $\tilde{\mathcal{X}}_{\text{pri}}$ is a stable morphism whose domain is smooth; a generic element of $\tilde{\mathcal{X}}_\mu$ of $\mu = (d_1, \dots, d_\ell)$ is a stable morphism whose domain is ℓ rational curves attached to a smooth elliptic curve so that the stable morphism is constant along the elliptic curve and has degree d_i along the i -th attached rational curve.

The corresponding stack $\tilde{\mathcal{Y}}$ has similar structure. First, we have the induced projection

$$\mathbf{p} : \tilde{\mathcal{Y}} \longrightarrow \tilde{\mathcal{X}}.$$

We let $\tilde{\mathcal{Y}}_\alpha = \tilde{\mathcal{Y}} \times_{\tilde{\mathcal{X}}} \tilde{\mathcal{X}}_\alpha$, where $\alpha = \text{pri}$ or $\mu \vdash d$; we denote $\tilde{\mathcal{X}}_{\text{gst}} = \cup_{\mu \vdash d} \tilde{\mathcal{X}}_\mu \subset \tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}_{\text{gst}} = \cup_{\mu \vdash d} \tilde{\mathcal{Y}}_\mu (= \tilde{\mathcal{Y}} \times_{\tilde{\mathcal{X}}} \tilde{\mathcal{X}}_{\text{gst}})$. Thus

$$(3.1) \quad \tilde{\mathcal{X}} = \tilde{\mathcal{X}}_{\text{pri}} \cup \tilde{\mathcal{X}}_{\text{gst}} \quad \text{and} \quad \tilde{\mathcal{Y}} = \tilde{\mathcal{Y}}_{\text{pri}} \cup \tilde{\mathcal{Y}}_{\text{gst}}.$$

We group the property of this decomposition as follows.

Proposition 3.1. *For any closed $y \in \tilde{\mathcal{Y}}$, we can find an étale neighborhood $\tilde{X} \rightarrow \tilde{\mathcal{X}}$ of $x = \mathbf{p}(y) \in \tilde{\mathcal{X}}$, an embedding \tilde{X} into a smooth affine scheme Z , and an embedding $\tilde{Y} := \tilde{X} \times_{\tilde{\mathcal{X}}} \tilde{\mathcal{Y}} \rightarrow Z' := Z \times \mathbb{A}^1$, such that*

- (1) let $p_Z : Z' = Z \times \mathbb{A}^1 \rightarrow Z$ be the first projection; then $\tilde{Y} \rightarrow \tilde{X}$ commutes with $p_Z : Z' \rightarrow Z$;
- (2) there are regular functions z_1, \dots, z_4 and $w_\mu \in \Gamma(\mathcal{O}_Z)$, $\mu \vdash d$, such that all subschemes $(z_i = 0)$, and $(w_\mu = 0) \subset Z$ are smooth; further, let $w = \prod_\mu w_\mu$, then $(z_1 \cdots z_4 \cdot w = 0) \subset Z$ has normal crossing singularities;
- (3) there is a smooth morphism $Z \rightarrow \mathcal{M}^w$ so that the composite $\tilde{Y} \rightarrow Z \rightarrow \mathcal{M}^w$ is identical to the composite $\tilde{Y} \rightarrow \tilde{\mathcal{Y}} \rightarrow \mathcal{M}^w$; further, locally the functions w_μ are pullbacks of functions on \mathcal{M}^w ;
- (4) let $t \in \Gamma(\mathcal{O}_{Z'})$ be the pull back of the standard coordinate function of \mathbb{A}^1 via $p_{\mathbb{A}^1} : Z' \rightarrow \mathbb{A}^1$; then as subschemes of Z' , $\tilde{Y} = (w \cdot z_{1 \leq i \leq 4} = w \cdot t = 0)^1$; $\tilde{Y}_{\text{pri}} = (z_{1 \leq i \leq 4} = t = 0)$; $\tilde{Y}_\mu = (w_\mu = 0)$, and $\tilde{Y}_{\text{gst}} = (w = 0)$;
- (5) identify $Z \cong Z' \cap (t = 0)$; then as subschemes of Z , $\tilde{X} = \tilde{Y} \cap (t = 0)$; $\tilde{X}_{\text{pri}} = \tilde{Y}_{\text{pri}}$, and $\tilde{X}_\mu = \tilde{Y}_\mu \cap (t = 0)$;

Proof. Recall that the proof of the structure result of $\tilde{\mathcal{X}}$ carried out in [HL1] is by deriving the local defining equation of \mathcal{X} , which was originally derived by Zinger using analytic method. The derivation of such defining equation relies on solving the following resolution problem: Let S be a scheme and $p_S : \mathcal{C}_S \rightarrow S$ be a flat family of nodal arithmetic genus one curves together with an invertible sheaf \mathcal{L}_S on \mathcal{C}_S so that its restriction to closed fibers \mathcal{C}_s , $s \in S$, are generated by global sections; one then finds an explicit complex of locally free sheaves of \mathcal{O}_S -modules that is quasi-isomorphic to $R^\bullet p_{S*} \mathcal{L}_S$.

Once such explicit complex is found, using that deforming a stable map $u : C \rightarrow \mathbb{P}^4$ amounts to deforming the data (C, L, u_1, \dots, u_5) (up to equivalence) of a pair of a line bundle on a curve and sections $u_i \in \Gamma(C, L)$, we obtain the local defining equation of \mathcal{X} (cf. [HL1]).

Deriving the resolution property of $R^\bullet p_{S*} \mathcal{L}_S$ is an elementary problem in algebraic geometry. It was done in [HL1] as follows: Suppose S as before is a scheme; $s_0 \in S$ is a closed point so that $\mathcal{C}_{s_0} = E \cup R$ is a union of a smooth elliptic curve and a smooth rational curve, and $\mathcal{L}_S|_E \cong \mathcal{O}_E$ and $\mathcal{L}_S|_R$ is ample. Let $\xi \in \Gamma(\mathcal{O}_S)$ be such that $(\xi = 0)$ is the locus where \mathcal{C}_S is nodal. Then after shrinking S if necessary, we have quasi-isomorphism of derived objects

$$R^\bullet p_{S*} \mathcal{L}_S \cong_{q.i.} [\mathcal{O}_S \xrightarrow{\times \xi} \mathcal{O}_S] \oplus [\mathcal{O}_S^{\oplus r} \longrightarrow 0],$$

where $r = \deg \mathcal{L}|_{\mathcal{C}_{s_0}}$.

We now derive the local defining equation of \mathcal{Y} . Since deforming a closed point $([u, C], \psi) \in \mathcal{Y}$ is equivalent to deforming (C, L, u_i, ψ) up to equivalence, where (C, L, u_i) is as before and $\psi \in \Gamma(C, L^{\otimes(-5)} \otimes \omega_C)$, we need to have, for the family $(\mathcal{C}_S \rightarrow S, \mathcal{L}_S)$ as before, the resolution property of $R^\bullet p_{S*} (\mathcal{L}_S^{\oplus 5} \oplus \mathcal{L}_S^{\otimes(-5)} \otimes \omega_{\mathcal{C}_S/S})$. Like the case for $R^\bullet p_{S*} \mathcal{L}_S$, this case can be derived from the simple case where $\mathcal{C}_{s_0} = E \cup R$ and $\mathcal{L}_S|_E \cong \mathcal{O}_E$ as before. In this case, by [HL1],

$$R^\bullet p_{S*} \mathcal{L}_S^{\otimes 5} \cong_{q.i.} [\mathcal{O}_S \xrightarrow{\times \xi} \mathcal{O}_S] \oplus [\mathcal{O}_S^{\oplus 5r} \longrightarrow 0].$$

By Serre duality, we obtain

$$R^\bullet p_{S*} \mathcal{L}_S^{\otimes(-5)} \otimes \omega_{\mathcal{C}_S/S} \cong_{q.i.} [\mathcal{O}_S \xrightarrow{\times \xi} \mathcal{O}_S] \oplus [0 \longrightarrow \mathcal{O}_S^{\oplus 5r}].$$

¹We use $w \cdot z_{1 \leq i \leq 4} = 0$ to mean $w \cdot z_1 = \dots = w \cdot z_4 = 0$.

Since it is a direct sum of $[\mathcal{O}_S \xrightarrow{\times \xi} \mathcal{O}_S]$ with a shift of a locally free sheaf, the proof of [HL1] can be adopted line by line to the case of \mathcal{Y} to prove the statement of the Lemma. Since the details are parallel, we will omit the details here. \square

Corollary 3.2. *All $\tilde{\mathcal{X}}_\alpha$ and $\tilde{\mathcal{Y}}_\alpha$, where $\alpha = \text{pri}$ or $\mu \vdash d$, are smooth; the tautological projection $\tilde{\mathcal{Y}}_\alpha \rightarrow \tilde{\mathcal{X}}_\alpha$ is an isomorphism (resp. an \mathbb{A}^1 -bundle) for $\alpha = \text{pri}$ (resp. $\alpha = \mu \vdash d$).*

Lemma 3.3. *Let $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} \subset h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})$ be the intrinsic normal cone embedded via the obstruction theory $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ of $\tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{D}}$.*

- (1) *Away from $\tilde{\mathcal{Y}}_{\text{gst}}$, it is the zero section of $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\text{gst}}}$.*
- (2) *Away from $\tilde{\mathcal{Y}}_{\text{pri}}$, it is a rank two subbundle stack of $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\text{pri}}}$.*

Proof. We consider the morphism

$$p : \tilde{\mathcal{Y}} \longrightarrow \tilde{\mathcal{D}}.$$

By the previous Proposition and Corollary, we know that both $\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\text{gst}}$ and the tautological morphism $\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\text{gst}} \rightarrow \mathcal{M}^w$ are smooth. Since the morphism $f_{\tilde{\mathcal{Y}}} : \mathcal{C}_{\tilde{\mathcal{Y}}} \rightarrow \mathbb{P}^4$ has positive degree along the (minimal) genus one subcurves of the fibers of $\mathcal{C}_{\tilde{\mathcal{Y}}}|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\text{gst}}} \rightarrow \tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\text{gst}}$, one checks that $p|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\text{gst}}} : \tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\text{gst}} \rightarrow \tilde{\mathcal{D}}$ is also smooth. Therefore, $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\text{gst}}}$ is the zero section of $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\text{gst}}}$.

For $\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\text{pri}}$, by the previous Proposition and Corollary, the tautological $\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\text{pri}} \rightarrow \mathcal{M}^w$ has its image a locally closed Cartier divisor in \mathcal{M}^w , and $\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\text{pri}}$ is smooth onto this image. Further, because the restriction of the tautological line bundle $\mathcal{L}_{\tilde{\mathcal{Y}}}$ to the minimal genus one subcurves of the fibers of $\mathcal{C}_{\tilde{\mathcal{Y}}}|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\text{pri}}} \rightarrow \tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\text{pri}}$ are generated by global sections and have degree zero, they are trivial line bundles. Thus the image $p(\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\text{pri}}) \subset \tilde{\mathcal{D}}$ is a locally closed local complete intersection codimension two substack, and $p|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\text{pri}}}$ is smooth onto this image. Thus $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\text{pri}}}$ is a rank two bundle stack over $\tilde{\mathcal{Y}} - \tilde{\mathcal{Y}}_{\text{pri}}$. \square

By this Lemma, we know that to each $\alpha = \text{pri}$ of $\mu \vdash d$, the cone $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ contains a unique irreducible component (of multiplicity one) that dominates $\tilde{\mathcal{Y}}_\alpha$. For the irreducible component that dominates $\tilde{\mathcal{Y}}_{\text{pri}}$, it is the closure of the zero section of $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}-\tilde{\mathcal{Y}}_{\text{gst}}}$ (in $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}_{\text{pri}}}$); we denote it by \mathbf{C}_{pri} . For $\mu \vdash d$, it is the closure of a rank two subbundle stack in $h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}_\mu}$ described in Lemma 3.3; we denote this subcone by \mathbf{C}'_μ .

There are possibly other irreducible components of $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ lying over $\tilde{\Delta} = \tilde{\mathcal{Y}}_{\text{pri}} \cap \tilde{\mathcal{Y}}_{\text{gst}}$. We group them into $\sum_{\mu \vdash d} \mathbf{C}''_\mu$ such that \mathbf{C}''_μ lies over $\tilde{\Delta}_\mu = \tilde{\Delta} \times_{\tilde{\mathcal{Y}}} \tilde{\mathcal{Y}}_\mu$. We comment that this grouping is not unique. We write $\mathbf{C}_\mu = \mathbf{C}'_\mu \cup \mathbf{C}''_\mu$.

Therefore

$$(3.2) \quad [\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}] = [\mathbf{C}_{\text{pri}}] + \sum_{\mu \vdash d} [\mathbf{C}_\mu] \in Z_* h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}).$$

Consequently, (denoting $[\mathbf{C}_{\text{gst}}] = \sum_{\mu \vdash d} [\mathbf{C}_\mu]$),

$$(3.3) \quad [\tilde{\mathcal{Y}}]_{\text{loc}}^{\text{vir}} = 0_{\tilde{\sigma}, \text{loc}}^! [\mathbf{C}_{\text{pri}}] + 0_{\tilde{\sigma}, \text{loc}}^! [\mathbf{C}_{\text{gst}}] = 0_{\tilde{\sigma}, \text{loc}}^! [\mathbf{C}_{\text{pri}}] + \sum_{\mu \vdash d} 0_{\tilde{\sigma}, \text{loc}}^! [\mathbf{C}_\mu].$$

Let $N_1(d)_Q^{\text{red}}$ be the reduced genus one GW-invariants of the quintic Q introduced in [LZ] (cf. Introduction).

Proposition 3.4. *We have $\deg 0_{\tilde{\sigma}, \text{loc}}^1[\mathbf{C}_{\text{pri}}] = (-1)^{5d} N_1(d)_Q^{\text{red}}$.*

Let (d) be the partition of d into a single part; i.e., the non-partition of d .

Proposition 3.5. *For $\mu \neq (d) \vdash d$, we have $\deg 0_{\tilde{\sigma}, \text{loc}}^1[\mathbf{C}_{\mu}] = 0$.*

Proposition 3.6. *We have $\deg 0_{\tilde{\sigma}, \text{loc}}^1[\mathbf{C}_{(d)}] = \frac{(-1)^{5d}}{12} N_0(d)_Q$.*

These three Propositions, the identity (3.3) and Proposition 2.3 combined give an algebraic proof of the hyperplane property of genus one GW-invariants of quintics proved originally via analytic method in [LZ, VZ, Zi1].

4. CONTRIBUTION FROM THE PRIMARY COMPONENT

Let $(f_{\tilde{\mathcal{Y}}}, \pi_{\tilde{\mathcal{Y}}}) : \mathcal{C}_{\tilde{\mathcal{Y}}} \rightarrow \mathbb{P}^4 \times \tilde{\mathcal{Y}}$ with $\psi_{\tilde{\mathcal{Y}}} \in \Gamma(\mathcal{C}_{\tilde{\mathcal{Y}}}, \mathcal{P}_{\tilde{\mathcal{Y}}})$ be the tautological family of $\tilde{\mathcal{Y}}$, where $\mathcal{L}_{\tilde{\mathcal{Y}}} = f_{\tilde{\mathcal{Y}}}^* \mathcal{O}(1)$ and $\mathcal{P}_{\tilde{\mathcal{Y}}} = \mathcal{L}_{\tilde{\mathcal{Y}}}^{\otimes (-5)} \otimes \omega_{\mathcal{C}_{\tilde{\mathcal{Y}}}/\tilde{\mathcal{Y}}}$. Recall that (the deformation complex of the relative obstruction theory) $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} = R^{\bullet} \pi_{\tilde{\mathcal{Y}*}}(\mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5} \oplus \mathcal{P}_{\tilde{\mathcal{Y}}})$. We let

$$\mathbf{H}_1 = h^1/h^0((R^{\bullet} \pi_{\tilde{\mathcal{Y}*}} \mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5})|_{\tilde{\mathcal{Y}}_{\text{pri}}}) \quad \text{and} \quad \mathbf{H}_2 = h^1/h^0((R^{\bullet} \pi_{\tilde{\mathcal{Y}*}} \mathcal{P}_{\tilde{\mathcal{Y}}})|_{\tilde{\mathcal{Y}}_{\text{pri}}}).$$

By the base change property of the h^1/h^0 -construction, we have

$$h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}) \times_{\tilde{\mathcal{Y}}} \tilde{\mathcal{Y}}_{\text{pri}} = \mathbf{H}_1 \times_{\tilde{\mathcal{Y}}_{\text{pri}}} \mathbf{H}_2.$$

Now let $\tilde{\mathcal{Y}}_{\text{pri}}^{\circ} = \tilde{\mathcal{Y}}_{\text{pri}} - \tilde{\Delta}$, where $\tilde{\Delta} = \tilde{\mathcal{Y}}_{\text{gst}} \cap \tilde{\mathcal{Y}}_{\text{pri}}$. By its definition and an easy vanishing argument, we have $(R^1 \pi_{\tilde{\mathcal{Y}*}} \mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5})|_{\tilde{\mathcal{Y}}_{\text{pri}}^{\circ}} = 0$, and $(R^1 \pi_{\tilde{\mathcal{Y}*}} \mathcal{P}_{\tilde{\mathcal{Y}}})|_{\tilde{\mathcal{Y}}_{\text{pri}}^{\circ}} \cong ((\pi_{\tilde{\mathcal{Y}*}} f_{\tilde{\mathcal{Y}}}^* \mathcal{O}(5))|_{\tilde{\mathcal{Y}}_{\text{pri}}^{\circ}})^{\vee}$, which is locally free of rank $5d$ over $\tilde{\mathcal{Y}}_{\text{pri}}^{\circ}$. We let $0_{\mathbf{H}_2} \subset \mathbf{H}_2$ be the closure of the zero section of $\mathbf{H}_2|_{\tilde{\mathcal{Y}}_{\text{pri}}^{\circ}}$ (in \mathbf{H}_2).

Lemma 4.1. *Let $\iota : \mathbf{H}_1 \times_{\tilde{\mathcal{Y}}_{\text{pri}}} \mathbf{H}_2 \rightarrow h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})$ be the inclusion, then we have*

$$[\mathbf{C}_{\text{pri}}] = \iota_*[\mathbf{H}_1 \times_{\tilde{\mathcal{Y}}_{\text{pri}}} 0_{\mathbf{H}_2}] \in Z_* h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}).$$

Proof. First, over $\tilde{\mathcal{Y}}_{\text{pri}}^{\circ}$, \mathbf{H}_1 is the zero bundle-stack. Thus by Lemma 3.3 and the discussion afterwards, the identity stated in the Lemma holds when restricted to $\tilde{\mathcal{Y}}_{\text{pri}}^{\circ}$. By their constructions, both \mathbf{C}_{pri} and $\mathbf{H}_1 \times_{\tilde{\mathcal{Y}}_{\text{pri}}} 0_{\mathbf{H}_2}$ are irreducible. Thus the identity holds. \square

Corollary 4.2. *We have $0_{\tilde{\sigma}, \text{loc}}^1([\mathbf{C}_{\text{pri}}]) = 0^!([0_{\mathbf{H}_2}])$, where the first is taken under the localized Gysin map of $A_* h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})_{\tilde{\sigma}}$, and the second is taken under the ordinary Gysin map of \mathbf{H}_2 .*

Proof. Since $\tilde{\mathcal{Y}}_{\text{pri}}$ is proper, and since the cosection localized Gysin map is the same as the ordinary Gysin map over proper bases, the proposition follows from that $0^![\mathbf{H}_1 \times_{\tilde{\mathcal{Y}}_{\text{pri}}} 0_{\mathbf{H}_2}] = 0^![0_{\mathbf{H}_2}]$, where the first $0^!$ is taken in $\mathbf{H}_1 \times_{\tilde{\mathcal{Y}}_{\text{pri}}} \mathbf{H}_2$ and second $0^!$ is taken in \mathbf{H}_2 . \square

To proceed, we prove a useful result.

Lemma 4.3. *Let $R = [\mathcal{R}_0 \rightarrow \mathcal{R}_1]$ be a complex of locally free sheaves on an integral Deligne-Mumford stack M such that $H^1(R)$ is a torsion sheaf on M and the image sheaf of $\mathcal{R}_0 \rightarrow \mathcal{R}_1$ is locally free. Let $U \subset M$ be the complement of the support of $H^1(R)$, and let $\mathbf{B} \subset h^1/h^0(R^\vee[-1])$ be the closure of the zero section of the vector bundle $h^1/h^0(R^\vee[-1]|_U) = H^0(R|_U)^\vee$. Then*

$$0^![\mathbf{B}] = e(H^0(R)^\vee) \in A_*M.$$

Proof. Let $\mathcal{K}_i = H^i(R)$, which fit into the exact sequence

$$(4.1) \quad 0 \longrightarrow \mathcal{K}_0 \xrightarrow{\alpha} \mathcal{R}_0 \xrightarrow{\beta} \mathcal{R}_1 \xrightarrow{\gamma} \mathcal{K}_1 \longrightarrow 0.$$

(Since \mathcal{K}_0 and \mathcal{R}_i are locally free, we will use the same symbol to denote its associated vector bundle.) By assumption and the choice of U , $\beta^\vee|_U$ is a sub-bundle homomorphism. Hence $[\mathcal{R}_0^\vee/\mathcal{R}_1^\vee]|_U$ is a vector bundle over U . Viewing $U \subset [\mathcal{R}_0^\vee/\mathcal{R}_1^\vee]|_U$ as the zero-section, its closure in the bundle stack $h^1/h^0(R^\vee[-1]|_U) = [\mathcal{R}_0^\vee/\mathcal{R}_1^\vee]$ is the \mathbf{B} referred to in the statement of the Lemma.

Let $b : \mathcal{R}_0^\vee \rightarrow [\mathcal{R}_0^\vee/\mathcal{R}_1^\vee]$ be the quotient. As $\beta^\vee(\mathcal{R}_1^\vee|_U) \subset \mathcal{R}_0^\vee|_U$ is a sub-bundle; we let \mathbf{D} be the closure of $\beta^\vee(\mathcal{R}_1^\vee|_U)$ in \mathcal{R}_0^\vee . The smoothness of b implies $b^{-1}(\mathbf{B}) = \mathbf{D}$. Hence

$$0^!([\mathbf{B}]) = 0^!([\mathbf{D}]) \in A_*M,$$

where the two Gysin maps are intersecting with the zero sections of $[\mathcal{R}_0^\vee/\mathcal{R}_1^\vee]$ and of \mathcal{R}_0^\vee , respectively.

We are left to show $0^!([\mathbf{D}]) = e(\mathcal{K}_0^\vee)$. (By assumption, \mathcal{K}_0 is locally free.) Since $\ker \gamma$ is locally free, $\alpha^\vee : \mathcal{R}_0^\vee \rightarrow \mathcal{K}_0^\vee$ is a surjection of vector bundles. On the other hand since $\mathcal{K}_1|_U = 0$ we can restrict (4.1) on U and take its dual to get an exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{R}_1^\vee|_U \xrightarrow{\beta^\vee} \mathcal{R}_0^\vee|_U \xrightarrow{\alpha^\vee} \mathcal{K}_0^\vee|_U \longrightarrow 0.$$

Hence the closure of $\beta^\vee(\mathcal{R}_1^\vee|_U)$ in \mathcal{R}_0^\vee equals the closure of $\ker(\alpha^\vee)|_U$ in \mathcal{R}_0^\vee ; thus $\mathbf{D} = \ker \alpha^\vee$. As $\mathcal{R}_0^\vee \rightarrow \mathcal{K}_0^\vee$ is a surjective bundle homomorphism, we have $0^!([\mathbf{D}]) = e(\mathcal{K}_0^\vee)$. \square

Let $(f_{\tilde{\mathcal{Y}}_{\text{pri}}}, \pi_{\tilde{\mathcal{Y}}_{\text{pri}}})$ be the restriction of $(f_{\tilde{\mathcal{Y}}}, \pi_{\tilde{\mathcal{Y}}})$ to $\tilde{\mathcal{Y}}_{\text{pri}}$. Let

$$R = (R^\bullet \pi_{\tilde{\mathcal{Y}}_*} f_{\tilde{\mathcal{Y}}}^* \mathcal{O}_{\mathbb{P}^4}(5))|_{\tilde{\mathcal{Y}}_{\text{pri}}} = R^\bullet \pi_{\tilde{\mathcal{Y}}_{\text{pri}}*} f_{\tilde{\mathcal{Y}}_{\text{pri}}}^* \mathcal{O}_{\mathbb{P}^4}(5).$$

By Serre duality, $R^\vee \cong (R^\bullet \pi_{\tilde{\mathcal{Y}}_*} f_{\tilde{\mathcal{Y}}}^* \mathcal{P}_{\tilde{\mathcal{Y}}})|_{\tilde{\mathcal{Y}}_{\text{pri}}}$. Like in [Beh, LT], we can represent $R^\bullet \pi_{\tilde{\mathcal{Y}}_{\text{pri}}*} f_{\tilde{\mathcal{Y}}_{\text{pri}}}^* \mathcal{O}_{\mathbb{P}^4}(5) = [\mathcal{R}_0 \rightarrow \mathcal{R}_1]$ as a complex of locally free sheaves. On the other hand, by [VZ] (see also [HL1], or Prop. 3.1), $\pi_{\tilde{\mathcal{Y}}_{\text{pri}}*} f_{\tilde{\mathcal{Y}}_{\text{pri}}}^* \mathcal{O}_{\mathbb{P}^4}(5)$ is locally free of rank $5d$, and $R^1 \pi_{\tilde{\mathcal{Y}}_{\text{pri}}*} f_{\tilde{\mathcal{Y}}_{\text{pri}}}^* \mathcal{O}(5)$ is torsion. Thus applying Corollary 4.2 and Lemma 4.3, we obtain

Corollary 4.4. *We have identity*

$$\deg 0_{\sigma, \text{loc}}^![\mathbf{C}_{\text{pri}}] = c_{5d}((\pi_{\tilde{\mathcal{Y}}_{\text{pri}}*} f_{\tilde{\mathcal{Y}}_{\text{pri}}}^* \mathcal{O}_{\mathbb{P}^4}(5))^\vee) = (-1)^{5d} c_{5d}(\pi_{\tilde{\mathcal{Y}}_{\text{pri}}*} f_{\tilde{\mathcal{Y}}_{\text{pri}}}^* \mathcal{O}_{\mathbb{P}^4}(5)).$$

Note that the right hand side is the reduced genus one GW-invariants introduced in [LZ].

5. COSECTION LOCALIZED GYSIN MAP VIA COMPACTIFICATIONS

We begin with a general discussion of a special type of localized Gysin maps. Let M be a Deligne-Mumford stack, L be a line bundle on M , and V_1 and V_2 be two vector bundles on M . Suppose we are given two vector bundle homomorphisms²

$$(5.1) \quad \xi_1 : V_1 \otimes L \longrightarrow \mathcal{O}_M, \quad \text{and} \quad \xi_2 : V_2 \longrightarrow \mathcal{O}_M$$

such that ξ_1 is surjective. (Here \mathcal{O}_M is viewed as the rank one trivial line bundle on M .)

We let $\gamma : W = \text{Total}(L) \rightarrow M$ be the total space of L . The tautological (identity) section $\epsilon \in \Gamma(W, \gamma^*L)$ paired with $\gamma^*(\xi_1)$ defines a bundle homomorphism

$$(5.2) \quad \tilde{\xi}_1 := \gamma^*(\xi_1)(\cdot \otimes \epsilon) : \tilde{V}_1 := \gamma^*V_1 \longrightarrow \mathcal{O}_W.$$

We let $\tilde{\xi}_2 = \gamma^*(\xi_2) : \tilde{V}_2 := \gamma^*V_2 \rightarrow \mathcal{O}_W$.

Let

$$\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2) : \tilde{V} := \tilde{V}_1 \oplus \tilde{V}_2 \longrightarrow \mathcal{O}_W.$$

Since ξ_1 is surjective, $\tilde{\xi}_1$ is surjective away from the zero-section $0_W = M \subset W$; thus the non-surjective locus $D(\tilde{\xi})$ of $\tilde{\xi}$ is contained in the zero-section $0_W \subset W$. We let $U = W - D(\tilde{\xi})$, and form

$$\tilde{V}(\tilde{\xi}) = \tilde{V}|_{D(\tilde{\xi})} \cup \ker\{\tilde{\xi}|_U : \tilde{V}|_U \rightarrow \mathcal{O}_U\} \subset \tilde{V}.$$

The cosection $\tilde{\xi}$ defines a localized Gysin map (cf. [KL])

$$(5.3) \quad 0_{\tilde{\xi}, \text{loc}}^! : A_*\tilde{V}(\tilde{\xi}) \longrightarrow A_*D(\tilde{\xi}).$$

This localized Gysin map has a simple interpretation in the situation under consideration. We let

$$\bar{\gamma} : \bar{W} = \mathbf{P}(L \oplus \mathcal{O}_M) \longrightarrow M$$

be the obvious compactification of $W = \bar{W} - D_\infty$, where $D_\infty = \mathbf{P}(L \oplus 0)$ is a divisor of \bar{W} . We extend \tilde{V}_1 and \tilde{V}_2 to \bar{W} via

$$\bar{V}_1 = \bar{\gamma}^*V_1(-D_\infty) \quad \text{and} \quad \bar{V}_2 = \bar{\gamma}^*V_2.$$

We let $\bar{\xi}_2 = \bar{\gamma}^*\xi_2$, which is the extension of $\tilde{\xi}_2$. Because of the expression (5.1), $\tilde{\xi}_1$ extends to a homomorphism $\bar{\xi}_1 : \bar{V}_1 = \bar{\gamma}^*(-D_\infty) \rightarrow \mathcal{O}_{\bar{W}}$. Let

$$\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2) : \bar{V} := \bar{V}_1 \oplus \bar{V}_2 \longrightarrow \mathcal{O}_{\bar{W}}.$$

Because $\tilde{\xi}_1$ has the form (5.2) and ξ_1 is surjective, $\bar{\xi}_1$ is surjective along D_∞ ; thus so does $\bar{\xi}$. Consequently, the non-surjective locus of $\bar{\xi}$ and $\tilde{\xi}$ are identical; namely

$$(5.4) \quad D(\bar{\xi}) = D(\tilde{\xi}).$$

Lemma 5.1. *Let $\iota_! : Z_*\tilde{V}(\tilde{\xi}) \rightarrow Z_*\bar{V}$ be defined by sending a closed integral $[C] \in Z_*\tilde{V}(\tilde{\xi})$ to the cycle of its closure in \bar{V} : $\iota_![C] = [\bar{C}] \in Z_*\bar{V}$, and then extending to $Z_*\tilde{V}(\tilde{\xi})$ by linearity. Let $\tau : D(\tilde{\xi}) \rightarrow M$ be the inclusion. Then we have*

$$\bar{\gamma}_* \circ 0_{\bar{V}}^! \circ \iota_! = \tau_* \circ 0_{\tilde{\xi}, \text{loc}}^! : Z_*\tilde{V}(\tilde{\xi}) \longrightarrow A_*M.$$

²Bundle homomorphisms in this paper are possibly degenerate, (i.e. those induced by sheaf-homomorphisms of respective bundles).

Proof. We let $\bar{V}(\bar{\xi})$ be defined similar to $\tilde{V}(\tilde{\xi})$ with “ \sim ” replaced by “ $\bar{\sim}$ ”. Let $0_{\bar{\xi}, \text{loc}}^! : Z_*\bar{V}(\bar{\xi}) \rightarrow A_*D(\bar{\xi})$ be the localized Gysin map associated to $\bar{\xi}$. Let $\tau' : \bar{V}(\bar{\xi}) \rightarrow \bar{V}$ and $\tau'' : D(\bar{\xi}) \rightarrow \bar{W}$ be the inclusions. By [KL, Prop 1.3], we have the commutative square

$$\begin{array}{ccc} Z_*\bar{V}(\bar{\xi}) & \xrightarrow{0_{\bar{\xi}, \text{loc}}^!} & A_*D(\bar{\xi}) \\ \downarrow \tau' & & \downarrow \tau'' \\ Z_*\bar{V} & \xrightarrow{0_{\bar{V}}^!} & A_*\bar{W}. \end{array}$$

On the other hand, since $\bar{\xi}$ is an extension of $\tilde{\xi}$, the homomorphism $\iota_! : Z_*\tilde{V}(\tilde{\xi}) \rightarrow Z_*\bar{V}$ factors through $\iota'_! : Z_*\tilde{V}(\tilde{\xi}) \rightarrow Z_*\bar{V}(\bar{\xi})$. Composing, we obtain

$$\tau'' \circ 0_{\bar{\xi}, \text{loc}}^! \circ \iota'_! = 0_{\bar{V}}^! \circ \tau' \circ \iota'_! = 0_{\bar{V}}^! \circ \iota_!.$$

Since $\bar{\xi}$ is an extension of $\tilde{\xi}$ and $D(\bar{\xi}) = D(\tilde{\xi})$, tracing through the construction of the localized Gysin maps, we conclude $0_{\bar{\xi}, \text{loc}}^! \circ \iota'_! = 0_{\tilde{\xi}, \text{loc}}^!$. Composed with $\bar{\gamma}_* : A_*\bar{W} \rightarrow A_*M$, we obtain

$$\bar{\gamma}_* \circ 0_{\bar{V}}^! \circ \iota_! = \bar{\gamma}_* \circ \tau'' \circ 0_{\bar{\xi}, \text{loc}}^! \circ \iota'_! = \tau_* \circ 0_{\tilde{\xi}, \text{loc}}^! : Z_*\tilde{V}(\tilde{\xi}) \rightarrow A_*M.$$

This proves the Lemma. \square

Corollary 5.2. *Let $C \subset \tilde{V}(\tilde{\xi})$ be closed and integral; let $\bar{C} \subset \bar{V}$ be its closure, and let $\bar{C}_b = \bar{C} \cap (0 \oplus \bar{V}_2)$. We let $N_{\bar{C}_b} \bar{C}$ be the normal cone to \bar{C}_b in \bar{C} ; it is a cycle in \bar{V} . Then we have*

$$(5.5) \quad \deg 0_{\bar{\xi}, \text{loc}}^![C] = \deg 0_{\bar{V}}^![\bar{C}] = \deg 0_{\bar{V}}^![N_{\bar{C}_b} \bar{C}].$$

The intersection \bar{C}_b has an easy description in the homogeneous case. Since W is the total space of the line bundle L , we have the dilation morphism

$$(5.6) \quad \mathbf{m}_t : W \rightarrow W$$

defined by sending $x \in L|_{x'}$ (over $x' \in M$) to $tx \in L|_{x'} \subset W$. This defines a \mathbb{C}^* -action on W that preserves the morphism $W \rightarrow M$ with \mathbb{C}^* acting trivially on M ; its fixed locus is the zero section $0_W \subset W$.

We define $\Phi_{i,0}(t) : \tilde{V}_i \rightarrow \mathbf{m}_t^* \tilde{V}_i$ to be the homomorphism that keeps γ^*e invariant, where $\gamma : W \rightarrow M$ and $\gamma^*e \in V_i$; and for $k \in \mathbb{Z}$, we define

$$(5.7) \quad \Phi_{i,k}(t) = t^k \cdot \Phi_{i,0}(t) : \tilde{V}_i \rightarrow \mathbf{m}_t^* \tilde{V}_i.$$

Definition 5.3. *We say a closed integral $C \subset \tilde{V}$ homogeneous of weight (k_1, k_2) if it is invariant under $(\Phi_{1,k_1}(t), \Phi_{2,k_2}(t))$ for all $t \in \mathbb{C}^*$. We say a cycle $\alpha \in Z_*\tilde{V}$ is homogeneous of weight (k_1, k_2) if each of its integral components is homogeneous of weight (k_1, k_2) .*

We now investigate the intersection $\bar{C}_b = \bar{C} \cap (0 \oplus \bar{V}_2)$ near D_∞ . Since this is a local problem, by replacing M by its affine étale chart, we can assume $M = \text{Spec } A$ is affine, and $L \cong \mathcal{O}_M$ and $V_i \cong \mathcal{O}_M^{\oplus n_i}$ are trivialized. Using such trivializations, we have induced isomorphisms

$$(5.8) \quad \bar{W} = M \times \mathbb{P}^1, \quad \bar{V}_2 \cong (M \times \mathbb{P}^1) \times \mathbb{A}^{n_2},$$

and $D_\infty = M \times \{\infty\} \subset M \times \mathbb{P}^1$.

Lemma 5.4. *Let $M = \operatorname{Spec} A$, and L and V_i are trivialized as stated. Let $C \subset \tilde{V}$ be a homogeneous closed integral substack of weight $(0, 1)$. Then under (5.8), and for a $B \subset M \times \mathbb{A}^{n_2}$,*

$$\overline{C} \cap (0 \oplus \overline{V}_2)|_{\overline{W}-0_W} = B \times (\mathbb{P}^1 - 0).$$

Proof. Adding $V_1 \cong \mathcal{O}_M^{\oplus n_1}$, we have the induced isomorphism

$$\tilde{V} \cong (M \times \mathbb{P}^1) \times \mathbb{A}^{n_1} \times \mathbb{A}^{n_2}.$$

We let t (resp. $x = (x_i)$; resp. $y = (y_j)$) be the standard coordinate variables of $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ (resp. of \mathbb{A}^{n_1} ; resp. of \mathbb{A}^{n_2}). Because C is homogeneous of weight $(0, 1)$, the ideal of $C|_{W-0_W} \subset \overline{V}|_{W-0_W}$ is generated by elements

$$\{f(x_1, \dots, x_{n_1}, t^{-1}y_1, \dots, t^{-1}y_{n_2}) \mid f \in J \subset A[x, y]\},$$

where J is an ideal in the polynomial ring $A[x, y] = A[x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}]$.

We now pick a new trivialization of \overline{V}_1 over $\overline{W} - 0_W$. We let e_1, \dots, e_{n_1} be a basis of \overline{V}_1 that is the pullback of a basis of V_1 (over M). As t^{-1} extends to a regular function near D_∞ and has vanishing order one along D_∞ , $e_i = t^{-1} \cdot \epsilon_i$, $1 \leq i \leq n_1$, form a basis of $\overline{V}_1|_{\overline{W}-0_W}$.

We let

$$(5.9) \quad \overline{V}|_{\overline{W}-0_W} \cong M \times (\mathbb{P}^1 - 0) \times \mathbb{A}^{n_1} \times \mathbb{A}^{n_2}$$

be the isomorphism induced by the given trivializations of L and V_2 , and the new trivialization of \overline{V}_1 using the basis e_i . We let $x' = (x'_i)$ be the standard coordinate variables of \mathbb{A}^{n_1} in (5.9), then x_i and x'_i are related by $x_i = t^{-1}x'_i$. Thus in the coordinates (x'_i, y_j) , $C|_{W-0_W}$ is defined by the ideal generated by

$$(5.10) \quad \{f(t^{-1}x'_1, \dots, t^{-1}x'_{n_1}, t^{-1}y_1, \dots, t^{-1}y_{n_2}) \mid f \in J \subset A[x, y]\}.$$

Finally, since $C \subset \tilde{V}$ is a cone, J is a homogeneous ideal; thus the same ideal is also generated by

$$(5.11) \quad \{f(x'_1, \dots, x'_{n_1}, y_1, \dots, y_{n_2}) \mid f \in J \subset A[x, y]\}.$$

Therefore, $\overline{C} \cap \overline{V}|_{\overline{W}-0_W} \subset \overline{V}|_{\overline{W}-0_W}$ is defined by the ideal generated by (5.11). Thus $\overline{C} \cap \overline{V}|_{\overline{W}-0_W} = B' \times (\mathbb{P}^1 - 0)$ for a $B' \subset M \times \mathbb{A}^{n_1} \times \mathbb{A}^{n_2}$. Intersecting with $\overline{V}_2|_{\overline{W}-0_W}$ proves the Lemma. \square

We will show that the pair $\tilde{\mathcal{X}}_{\text{gst}} \subset \tilde{\mathcal{Y}}_{\text{gst}}$ fits into the description of the pair $M \subset W$ described. To this end, we introduce the corresponding vector bundles V_i and \tilde{V}_i . As the deformation complex of $\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}$ is $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} = R^\bullet \pi_{\tilde{\mathcal{Y}}}^*(\mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5} \oplus \mathcal{P}_{\tilde{\mathcal{Y}}})$, we introduce

$$\tilde{\mathbf{V}}_1 = h^1/h^0((R^\bullet \pi_{\tilde{\mathcal{Y}}}^* \mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5})|_{\tilde{\mathcal{Y}}_{\text{gst}}}), \quad \tilde{\mathbf{V}}_2 = h^1/h^0((R^\bullet \pi_{\tilde{\mathcal{Y}}}^* \mathcal{P}_{\tilde{\mathcal{Y}}})|_{\tilde{\mathcal{Y}}_{\text{gst}}}), \quad \tilde{\mathbf{V}} = \tilde{\mathbf{V}}_1 \times_{\tilde{\mathcal{Y}}_{\text{gst}}} \tilde{\mathbf{V}}_2.$$

By the base change property of the h^1/h^0 -construction, and the construction of \mathbf{C}_{gst} (cf. (3.3)),

$$[\mathbf{C}_{\text{gst}}] \in Z_* \tilde{\mathbf{V}}; \quad \tilde{\mathbf{V}} = h^1/h^0(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})|_{\tilde{\mathcal{Y}}_{\text{gst}}}.$$

We let

$$(5.12) \quad \tilde{V}_1 = H^1((R^\bullet \pi_{\tilde{\mathcal{Y}}}^* \mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5})|_{\tilde{\mathcal{Y}}_{\text{gst}}}), \quad \tilde{V}_2 = H^1((R^\bullet \pi_{\tilde{\mathcal{Y}}}^* \mathcal{P}_{\tilde{\mathcal{Y}}})|_{\tilde{\mathcal{Y}}_{\text{gst}}}), \quad \text{and} \quad \tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2.$$

Since both \tilde{V}_1 and \tilde{V}_2 are locally free on $\tilde{\mathcal{Y}}_{\text{gst}}$, by abuse of notation, we will view them as vector bundles as well. Then using the arrow $(R^\bullet \pi_{\tilde{\mathcal{Y}}}^* \mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5})|_{\tilde{\mathcal{Y}}_{\text{gst}}} \rightarrow$

$H^1((R^\bullet \pi_{\tilde{\mathcal{Y}}}^* \mathcal{L}_{\tilde{\mathcal{Y}}}^{\oplus 5})|_{\tilde{\mathcal{Y}}_{\text{gst}}})$, we obtain a canonical morphism $\rho_1 : \tilde{\mathbf{V}}_1 \rightarrow \tilde{V}_1$; similarly, we have a canonical $\rho_2 : \tilde{\mathbf{V}}_2 \rightarrow \tilde{V}_2$. Note that both ρ_i are proper, and \tilde{V}_i are (relative to $\tilde{\mathcal{Y}}_{\text{gst}}$) coarse moduli spaces of $\tilde{\mathbf{V}}_i$. By the definition of \tilde{V}_i , the sheaf of sections of \tilde{V} is $\mathcal{O}b_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{Y}}_{\text{gst}}}$. We denote by $\tilde{\xi}$ the restriction

$$(5.13) \quad \tilde{\xi} := \tilde{\sigma}|_{\tilde{\mathcal{Y}}_{\text{gst}}} : \tilde{V} \longrightarrow \mathcal{O}_{\tilde{\mathcal{Y}}_{\text{gst}}},$$

where $\tilde{\sigma}$ is given in (2.11).

We now justify the symbols for the pair $M \subset W$. Let $(f_{\tilde{\mathcal{X}}_{\text{gst}}}, \pi_{\tilde{\mathcal{X}}_{\text{gst}}}) : \mathcal{C}_{\tilde{\mathcal{X}}_{\text{gst}}} \rightarrow \mathbb{P}^4 \times \tilde{\mathcal{X}}_{\text{gst}}$ be the pullback of the universal family of $\overline{\mathcal{M}}_1(\mathbb{P}^4, d)$. Let $\mathcal{L}_{\tilde{\mathcal{X}}_{\text{gst}}} = f_{\tilde{\mathcal{X}}_{\text{gst}}}^* \mathcal{O}(1)$ and $\mathcal{P}_{\tilde{\mathcal{X}}_{\text{gst}}} = \mathcal{L}_{\tilde{\mathcal{X}}_{\text{gst}}}^{\otimes (-5)} \otimes \omega_{\mathcal{C}_{\tilde{\mathcal{X}}_{\text{gst}}}/\tilde{\mathcal{X}}_{\text{gst}}}$. Then by [HL1], $\tilde{\mathcal{Y}}_{\text{gst}}$ is the total space of a line bundle on $\tilde{\mathcal{X}}_{\text{gst}}$:

$$L = (\pi_{\tilde{\mathcal{X}}_{\text{gst}}})_* \mathcal{P}_{\tilde{\mathcal{X}}_{\text{gst}}}.$$

Thus letting $M = \tilde{\mathcal{X}}_{\text{gst}}$ and $W = \tilde{\mathcal{Y}}_{\text{gst}}$ fit into the description $W = \text{Total}(L)$ given before.

We next introduce V_i and ξ_i on M that fit the whole package as well:

$$(5.14) \quad V_1 = R^1 \pi_{\tilde{\mathcal{X}}_{\text{gst}}}^* \mathcal{L}_{\tilde{\mathcal{X}}_{\text{gst}}}^{\oplus 5}, \quad V_2 = R^1 \pi_{\tilde{\mathcal{X}}_{\text{gst}}}^* \mathcal{P}_{\tilde{\mathcal{X}}_{\text{gst}}}, \quad \text{and} \quad V = V_1 \oplus V_2.$$

They are vector bundles (locally free sheaves) on $\tilde{\mathcal{X}}_{\text{gst}} (= M)$. Let

$$\gamma : W := \tilde{\mathcal{Y}}_{\text{gst}} = \text{Total}(L) \longrightarrow \tilde{\mathcal{X}}_{\text{gst}} = M$$

be the induced (tautological) projection.

Lemma 5.5. *We have canonical isomorphisms $\gamma^* V_i = \tilde{V}_i$; further there are $\xi_1 : V_1 \otimes L \rightarrow \mathcal{O}_M$ and $\xi_2 : V_2 \rightarrow \mathcal{O}_M$ so that $\tilde{\xi}$ in (5.13) is identical to the $\tilde{\xi}$ constructed from (ξ_1, ξ_2) via the construction given in (5.2) and after.*

Proof. We let $f_{\tilde{\mathcal{Y}}_{\text{gst}}}, \pi_{\tilde{\mathcal{Y}}_{\text{gst}}}, \mathcal{C}_{\tilde{\mathcal{Y}}_{\text{gst}}}, \mathcal{L}_{\tilde{\mathcal{Y}}_{\text{gst}}}$ and $\mathcal{P}_{\tilde{\mathcal{Y}}_{\text{gst}}}$ be objects over $\tilde{\mathcal{Y}}_{\text{gst}}$ defined similarly as that over $\tilde{\mathcal{X}}_{\text{gst}}$. Then the \tilde{V}_1 and \tilde{V}_2 in (5.12) are $\tilde{V}_1 = R^1 \pi_{\tilde{\mathcal{Y}}_{\text{gst}}}^* \mathcal{L}_{\tilde{\mathcal{Y}}_{\text{gst}}}^{\oplus 5}$ and $\tilde{V}_2 = R^1 \pi_{\tilde{\mathcal{Y}}_{\text{gst}}}^* \mathcal{P}_{\tilde{\mathcal{Y}}_{\text{gst}}}$. On the other hand, since we have canonical isomorphism $\mathcal{C}_{\tilde{\mathcal{Y}}_{\text{gst}}} \cong \mathcal{C}_{\tilde{\mathcal{X}}_{\text{gst}}} \times_{\tilde{\mathcal{X}}_{\text{gst}}} \tilde{\mathcal{Y}}_{\text{gst}}$, letting $\tilde{\gamma} : \mathcal{C}_{\tilde{\mathcal{Y}}_{\text{gst}}} \rightarrow \mathcal{C}_{\tilde{\mathcal{X}}_{\text{gst}}}$ be the induced projection, then we obtain $\tilde{\gamma}^* \mathcal{L}_{\tilde{\mathcal{Y}}_{\text{gst}}} = \mathcal{L}_{\tilde{\mathcal{X}}_{\text{gst}}}$, and same for $\mathcal{P}_{\tilde{\mathcal{Y}}_{\text{gst}}}$. Applying the base change formula, (since $R^2 \pi_{\tilde{\mathcal{X}}_{\text{gst}}}^* = 0$.) we obtain the canonical isomorphisms

$$\tilde{V}_1 = R^1 \pi_{\tilde{\mathcal{Y}}_{\text{gst}}}^* \mathcal{L}_{\tilde{\mathcal{Y}}_{\text{gst}}}^{\oplus 5} = R^1 \pi_{\tilde{\mathcal{Y}}_{\text{gst}}}^* \tilde{\gamma}^* \mathcal{L}_{\tilde{\mathcal{X}}_{\text{gst}}}^{\oplus 5} \cong \gamma^* R^1 \pi_{\tilde{\mathcal{X}}_{\text{gst}}}^* \mathcal{L}_{\tilde{\mathcal{X}}_{\text{gst}}}^{\oplus 5} = \gamma^* V_1;$$

the same reason gives a canonical $\tilde{V}_2 \cong \gamma^* V_2$.

For the $\tilde{\xi}$ defined in (5.13), we recall the construction of the cosection σ in (1.3). For any closed $y \in \tilde{\mathcal{Y}}_{\text{gst}}$ over $x \in \tilde{\mathcal{X}}_{\text{gst}}$ associated to $[u, C]$ with $u = (u_i) \in H^0(u^* \mathcal{O}(1)^{\oplus 5})$ and $\psi \in H^0(u^* \mathcal{O}(-5) \otimes \omega_C)$, $\tilde{V}_1|_y = H^1(u^* \mathcal{O}(1)^{\oplus 5})$, $\tilde{V}_2|_y = H^1(u^* \mathcal{O}(-5) \otimes \omega_C)$, and

$$\tilde{\sigma}(y) : \tilde{V}_1|_y \oplus \tilde{V}_2|_y \longrightarrow \mathbb{C}; \quad (\dot{u}_i, \dot{\psi}) \mapsto 5\psi \sum_{i=1}^5 u_i^4 \dot{u}_i + \dot{\psi} \sum_{i=1}^5 u_i^5.$$

Accordingly, we define $\xi_1 : V_1 \otimes L \rightarrow \mathcal{O}_{\tilde{\mathcal{X}}_{\text{gst}}}$ and $\xi_2 : V_2 \rightarrow \mathcal{O}_{\tilde{\mathcal{X}}_{\text{gst}}}$ over $x \in \tilde{\mathcal{X}}_{\text{gst}}$ to be

$$\sigma_1(x)((\dot{u}_i) \otimes \psi) = 5\psi \sum u_i^4 \dot{u}_i, \quad \text{and} \quad \xi_2(x)(\dot{\psi}) = \dot{\psi} \sum u_i^5.$$

With this definition, by [CL1, Prop 3.4], ξ_1 is surjective. As $\tilde{\xi}$ is constructed following (5.13) that is the restriction of $\tilde{\sigma}$ over $\tilde{\mathcal{Y}}_{\text{gst}}$, this proves the Lemma. \square

In the following, we will view V_i , \tilde{V}_i and $\tilde{\xi}$ be defined in (5.12), (5.13) and (5.14), which also satisfy the properties specified at and before (5.2).

Following [CL1], the non-surjective locus $D(\tilde{\xi})$ of $\tilde{\xi} = \tilde{\sigma}|_{\tilde{\mathcal{Y}}_{\text{gst}}}$ is

$$D(\tilde{\sigma}) \times_{\tilde{\mathcal{X}}} \tilde{\mathcal{X}}_{\text{gst}} = \overline{M}_1(Q, d) \times_{\overline{M}_1(\mathbb{P}^4, d)} \tilde{\mathcal{X}}_{\text{gst}},$$

which is proper. As before, we form the substack $\tilde{V}(\tilde{\xi}) \subset \tilde{V}$; we let C_{gst} be the coarse moduli of \mathbf{C}_{gst} relative to $\tilde{\mathcal{Y}}_{\text{gst}}$, thus $C_{\text{gst}} \subset \tilde{V}$ since \tilde{V} is the coarse moduli of $\tilde{\mathbf{V}}$. Further, since the projection $\rho := \rho_1 \times \rho_2 : \tilde{\mathbf{V}} \rightarrow \tilde{V}$ is smooth, we have an identity of cycles $\rho^*[C_{\text{gst}}] = [\mathbf{C}_{\text{gst}}] \in Z_*\tilde{\mathbf{V}}$. Finally, because $[\mathbf{C}_{\text{gst}}] \in Z_*\tilde{\mathbf{V}}(\tilde{\sigma})$ (cf. [CL1]), we have

$$[C_{\text{gst}}] \in Z_*\tilde{V}(\tilde{\xi}).$$

As before, $0_{\tilde{\xi}, \text{loc}}^! : A_*\tilde{V}(\tilde{\xi}) \rightarrow A_*D(\tilde{\xi})$ is the localized Gysin map.

Proposition 5.6. *We have the identity*

$$0_{\tilde{\sigma}, \text{loc}}^![\mathbf{C}_{\text{gst}}] = 0_{\tilde{\xi}, \text{loc}}^![\mathbf{C}_{\text{gst}}] \in A_*D(\tilde{\xi}).$$

Proof. We can find a vector bundle F on $\tilde{\mathcal{Y}}_{\text{gst}}$ and a surjective bundle map $\phi : F \rightarrow \tilde{\mathbf{V}}$. Let $\tilde{\sigma}_\phi : F \rightarrow \mathcal{O}_{\tilde{\mathcal{Y}}_{\text{gst}}}$ be the pullback cosection. Then $D(\tilde{\sigma}_\phi) = D(\tilde{\xi})$, $\phi^*[\mathbf{C}_{\text{gst}}] \in F(\tilde{\sigma}_\phi)$, and

$$0_{\tilde{\sigma}_\phi, \text{loc}}^!(\phi^*[\mathbf{C}_{\text{gst}}]) = 0_{\tilde{\sigma}, \text{loc}}^![\mathbf{C}_{\text{gst}}] \in A_*D(\tilde{\xi}).$$

On the other hand, let $\bar{\phi} : F \rightarrow \tilde{V}$ be the composition of ϕ with ρ , then $\tilde{\sigma}_\phi = \bar{\phi}^*(\tilde{\xi})$ and $\bar{\phi}^*[C_{\text{gst}}] = \phi^*[\mathbf{C}_{\text{gst}}]$. Thus

$$0_{\tilde{\sigma}_\phi, \text{loc}}^!(\phi^*[\mathbf{C}_{\text{gst}}]) = 0_{\bar{\phi}^*(\tilde{\xi}), \text{loc}}^!(\bar{\phi}^*[C_{\text{gst}}]) = 0_{\tilde{\xi}, \text{loc}}^![\mathbf{C}_{\text{gst}}] \in A_*D(\tilde{\xi}).$$

This proves the Proposition. \square

Finally, we verify

Lemma 5.7. *The substack $C_{\text{gst}} \subset \tilde{V}$ is homogeneous of weight $(0, 1)$.*

Proof. As the obstruction theory of $\tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{D}}$ is the pullback of that of $\mathcal{Y} \rightarrow \mathcal{D}$, and the later is via the open \mathcal{D} -embedding (cf. (2.1))

$$(5.15) \quad j : \mathcal{Y} = \overline{M}_1(\mathbb{P}^4, d)^p \xrightarrow{\subset} \mathfrak{S} := C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P})),$$

we will prove a corresponding result for \mathfrak{S} .

Let $\pi_{\mathfrak{S}} : \mathcal{C}_{\mathfrak{S}} \rightarrow \mathfrak{S}$, $\mathcal{L}_{\mathfrak{S}}$ and $\mathcal{P}_{\mathfrak{S}}$ be the pullback of $(\pi : \mathcal{C} \rightarrow \mathcal{D}, \mathcal{L}, \mathcal{P})$. The modular construction of \mathfrak{S} provides us a universal section

$$(5.16) \quad (u_{\mathfrak{S}, i}, \psi_{\mathfrak{S}}) \in \Gamma(\mathcal{C}_{\mathfrak{S}}, \mathcal{L}_{\mathfrak{S}}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{S}}) = \Gamma(\mathfrak{S}, \pi_{\mathfrak{S}*}(\mathcal{L}_{\mathfrak{S}}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{S}})), \quad (i = 1, \dots, 5).$$

Namely, each closed $\xi = (\mathcal{C}_{\xi}, u_{\xi, i}, \psi_{\xi}) \in \mathfrak{S}$ has $u_{\mathfrak{S}, i}|_{\xi} \equiv u_{\xi, i}$ and $\psi_{\mathfrak{S}}|_{\xi} \equiv \psi_{\xi}$.

For $t \in \mathbb{C}^*$, we introduce an automorphism

$$\mathbf{a}_t : \mathcal{L}^{\oplus 5} \oplus \mathcal{P} \longrightarrow \mathcal{L}^{\oplus 5} \oplus \mathcal{P}$$

that is $\text{id}_{\mathcal{L}^{\oplus 5}}$ when restricted to the summand $\mathcal{L}^{\oplus 5}$, and is $t \cdot \text{id}_{\mathcal{P}}$ when restricted to the summand \mathcal{P} . Clearly, \mathbf{a}_t commutes with the arrows of \mathcal{D} , and thus is a well-defined automorphism. The collection $\{\mathbf{a}_t \mid t \in \mathbb{C}^*\}$ defines a \mathbb{C}^* -action on $\mathcal{L}^{\oplus 5} \oplus \mathcal{P}$, where \mathbb{C}^* acts trivially on \mathcal{D} .

Following the moduli construction of \mathfrak{S} , the automorphism \mathbf{a}_t induces a \mathcal{D} -automorphism $\tilde{\mathbf{a}}_t : \mathfrak{S} \rightarrow \mathfrak{S}$: it is defined by sending $(\mathcal{C}_\tau, \mathcal{L}_\tau, u_i, \psi) \in \mathfrak{S}(T)$, where T is any scheme, to $(\mathcal{C}_\tau, \mathcal{L}_\tau, u_i, \psi)^t = (\mathcal{C}_\tau, \mathcal{L}_\tau, u_i, t \cdot \psi)$. The collection $\{\tilde{\mathbf{a}}_t \mid t \in \mathbb{C}^*\}$ forms a \mathbb{C}^* -action on \mathfrak{S} ; under this action the tautological projection $\mathfrak{S} \rightarrow \mathcal{D}$ is \mathbb{C}^* -equivariant.

Because $\mathcal{C}_\mathfrak{S} = \mathcal{C} \times_{\mathcal{D}} \mathfrak{S}$, and because $\mathfrak{S} \rightarrow \mathcal{D}$ is \mathbb{C}^* -equivariant (via $\tilde{\mathbf{a}}_t$ on \mathfrak{S} and the trivial \mathbb{C}^* -action on \mathcal{D}), the trivial \mathbb{C}^* -action on \mathcal{C} and the $\tilde{\mathbf{a}}_t$ on \mathfrak{S} lifts to a \mathbb{C}^* -action on $\mathcal{C}_\mathfrak{S} \rightarrow \mathfrak{S}$. We denote this action by $\varphi_0(t) : \mathcal{C}_\mathfrak{S} \rightarrow \tilde{\mathbf{a}}_t^* \mathcal{C}_\mathfrak{S}$. Then since $\mathcal{L}_\mathfrak{S}$ and $\mathcal{P}_\mathfrak{S}$ are pullback of \mathcal{L} and \mathcal{P} on \mathcal{C} , $\mathcal{L}_\mathfrak{S}$ and $\mathcal{P}_\mathfrak{S}$ admit the obvious \mathbb{C}^* -linearizations

$$(5.17) \quad \varphi_1(t) : \mathcal{L}_\mathfrak{S} \longrightarrow \varphi_0(t)^* \mathcal{L}_\mathfrak{S}, \quad \varphi_2(t) : \mathcal{P}_\mathfrak{S} \longrightarrow \varphi_0(t)^* \mathcal{P}_\mathfrak{S}$$

characterized by that their \mathbb{C}^* -equivariant sections are pullback sections of \mathcal{L} and \mathcal{P} , respectively.

The action $\tilde{\mathbf{a}}_t$ induces a tautological \mathbb{C}^* -action (linearization)

$$(5.18) \quad \bar{\varphi}_1(t) : \mathcal{L}_\mathfrak{S} \rightarrow \varphi_0(t)^* \mathcal{L}_\mathfrak{S}, \quad \bar{\varphi}_2(t) : \mathcal{P}_\mathfrak{S} \rightarrow \varphi_0(t)^* \mathcal{P}_\mathfrak{S}$$

characterized by that the universal section (5.16) is \mathbb{C}^* -equivariant under this linearization; i.e.,

$$(5.19) \quad (\bar{\varphi}_1(t)(u_{\mathfrak{S},i}), \bar{\varphi}_2(t)(\psi_\mathfrak{S})) = (\varphi_0(t)^*(u_{\mathfrak{S},i}), \varphi_0(t)^*(\psi_\mathfrak{S})).$$

By the construction of $\tilde{\mathbf{a}}_t$ and the two \mathbb{C}^* -actions, we see that

$$\bar{\varphi}_1(t) = \varphi_1(t) \quad \text{and} \quad \bar{\varphi}_2(t) = t \cdot \varphi_2(t).$$

Let $\phi_{\mathfrak{S}/\mathcal{D}} : L_{\mathfrak{S}/\mathcal{D}}^\bullet \rightarrow (E_{\mathfrak{S}/\mathcal{D}})^\vee$ be the perfect relative obstruction theory constructed in [CL1] by use of $(u_{\mathfrak{S},i}, \psi_\mathfrak{S})$. (Its pullback to \mathcal{Y} via $\mathcal{Y} \rightarrow \mathfrak{S}$ is the relative perfect obstruction theory of $\mathcal{Y} \rightarrow \mathcal{D}$.) Tracing through the construction of the obstruction theory $\phi_{\mathfrak{S}/\mathcal{D}}$ in [CL1, Prop 2.5], using that the universal section $(u_{\mathfrak{S},i}, \psi_\mathfrak{S})$ is equivariant under the tautological \mathbb{C}^* -linearization $(\bar{\varphi}_1, \bar{\varphi}_2)$, we conclude that the obstruction theory $\phi_{\mathfrak{S}/\mathcal{D}}$ is \mathbb{C}^* -equivariant with respect to the same \mathbb{C}^* -linearization.

We consider the composite

$$\tilde{j} : \tilde{\mathcal{Y}} \longrightarrow \mathcal{Y} \xrightarrow{j} \mathfrak{S}.$$

of the tautological $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ with the open embedding j in (5.15). Since the obstruction theory of $\tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{D}}$ is the pullback of that of $\mathfrak{S} \rightarrow \mathcal{D}$, we have canonical isomorphism

$$(5.20) \quad \tilde{j}^* E_{\mathfrak{S}/\mathcal{D}} \xrightarrow{\cong} E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}.$$

Secondly, since \tilde{j} is a \mathcal{D} -morphism and $\tilde{\mathcal{Y}}$ is constructed from $\mathcal{Y} \rightarrow \mathcal{D}$ via a base change $\tilde{\mathcal{D}} \rightarrow \mathcal{D}$, the \mathbb{C}^* -action on \mathfrak{S} lifts to a unique \mathbb{C}^* -action on $\tilde{\mathcal{Y}}$ that makes \tilde{j} \mathbb{C}^* -equivariant. Using (5.20), we endow $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ the \mathbb{C}^* -linearization induced based on $(\bar{\varphi}_1, \bar{\varphi}_2)$. This way, since $\phi_{\mathfrak{S}/\mathcal{D}}$ is \mathbb{C}^* -equivariant (via $(\bar{\varphi}_1, \bar{\varphi}_2)$), we conclude that $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ is \mathbb{C}^* -equivariant with the \mathbb{C}^* -linearization just introduced.

Finally, it is direct to check that the introduced \mathbb{C}^* -action on $\tilde{\mathcal{Y}}$ restricting to $\tilde{\mathcal{Y}}_{\text{gst}}$ is the \mathbb{C}^* -action \mathbf{m}_t constructed in (5.6); also, by [CL1, Prop 2.5], $E_{\mathfrak{S}/\mathcal{D}} = R^\bullet \pi_{\mathfrak{S}*}(\mathcal{L}_\mathfrak{S}^{\oplus 5} \oplus \mathcal{P}_\mathfrak{S})$, and the introduced \mathbb{C}^* -linearization on $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} \cong \tilde{j}^* E_{\mathfrak{S}/\mathcal{D}}$ restricting to $\tilde{\mathcal{Y}}_{\text{gst}}$ gives the linearization $(\Phi_{1,0}, \Phi_{2,1})$ of $\tilde{V}_1 \oplus \tilde{V}_2$, in the notation of (5.7).

Since $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ is \mathbb{C}^* -equivariant, the intrinsic normal cone $\mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} \subset h^1/h^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})$ is \mathbb{C}^* -equivariant; therefore C_{gst} is \mathbb{C}^* -equivariant under the \mathbb{C}^* -action $(\Phi_{1,0}, \Phi_{2,1})$, which is equivalent to say that C_{gst} is homogeneous of weight $(0, 1)$. This proves the Lemma. \square

6. PROOF OF PROPOSITION 3.5

We study the cone $C_{\text{gst}} \subset \tilde{V}$ in this section. Let $\Delta = \tilde{\mathcal{X}}_{\text{pri}} \cap \tilde{\mathcal{X}}_{\text{gst}}$, and recall

$$\tilde{\Delta} = \tilde{\mathcal{Y}} \times_{\tilde{\mathcal{X}}} \Delta = \tilde{\mathcal{Y}}_{\text{gst}} \times_{\tilde{\mathcal{X}}_{\text{gst}}} \Delta = W \times_M \Delta$$

is a line bundle over Δ . We continue to denote by $\gamma : \tilde{\mathcal{Y}}_{\text{gst}} \rightarrow \tilde{\mathcal{X}}_{\text{gst}}$ the projection.

Proposition 6.1. *We have $C_{\text{gst}} \cap (0 \oplus \tilde{V}_2)|_{\tilde{\mathcal{Y}}_{\text{gst}} - \tilde{\Delta}} = \tilde{\mathcal{Y}}_{\text{gst}} - \tilde{\Delta} \subset \tilde{\mathcal{Y}}_{\text{gst}}$; further, there is a sub-line bundle $\mathcal{F} \subset V_2|_{\Delta}$ so that, denoting $\tilde{\mathcal{F}} = \gamma^* \mathcal{F} \subset \tilde{V}_2|_{\tilde{\Delta}}$,*

$$C_{\text{gst}} \cap (0 \oplus \tilde{V}_2)|_{\tilde{\Delta}} \subset \tilde{\mathcal{F}} \subset \tilde{V}_2|_{\tilde{\Delta}}.$$

The proposition will be proved via a sequence of Lemmas. First, following the argument in [CL1, Sect 5.2], the relative obstruction theories of the triple $(\tilde{\mathcal{Y}}, \tilde{\mathcal{X}}, \tilde{\mathcal{D}})$ fit into a compatible diagram of distinguished triangles:

$$(6.1) \quad \begin{array}{ccccc} \gamma^*(E_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}})^\vee & \longrightarrow & (E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}})^\vee & \longrightarrow & (E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}})^\vee \xrightarrow{+1} \longrightarrow \\ \downarrow \gamma^* \phi_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}} & & \downarrow \phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} & & \downarrow \phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}} \\ \gamma^* L_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}}^\bullet & \longrightarrow & L_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}^\bullet & \longrightarrow & L_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}^\bullet \xrightarrow{+1} \longrightarrow \end{array}$$

Here $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$ is given by (2.10); $E_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}} = R^\bullet \pi_{\tilde{\mathcal{X}}}^* \mathcal{L}_{\tilde{\mathcal{X}}}^{\oplus 5}$ and $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}} = R^\bullet \pi_{\tilde{\mathcal{Y}}}^* \mathcal{P}_{\tilde{\mathcal{Y}}}$. (As usual, $\mathcal{L}_{\tilde{\mathcal{Y}}} = f_{\tilde{\mathcal{Y}}}^* \mathcal{O}(1)$ and $\mathcal{P}_{\tilde{\mathcal{Y}}} = \mathcal{L}_{\tilde{\mathcal{Y}}}^{\otimes (-5)} \otimes \omega_{\mathcal{C}_{\tilde{\mathcal{Y}}}/\tilde{\mathcal{Y}}}$.) Taking the cohomologies of the duals of the top row over $\tilde{\mathcal{Y}}_{\text{gst}}$, we obtain exact sequences of sheaves on $\tilde{\mathcal{Y}}_{\text{gst}}$:

$$(6.2) \quad \begin{array}{ccccccc} H^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}|_{\tilde{\mathcal{Y}}_{\text{gst}}}) & \longrightarrow & H^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{Y}}_{\text{gst}}}) & \longrightarrow & H^0(\gamma^* E_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}}|_{\tilde{\mathcal{Y}}_{\text{gst}}}) \\ \parallel & & \parallel & & \parallel \\ \tilde{V}_2 & \xrightarrow{\tilde{\beta}_1} & \tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2 & \xrightarrow{\tilde{\beta}_2} & \tilde{V}_1. \end{array}$$

Here the vertical identities are given by the explicit form of the complexes $E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$, etc.; $\tilde{\beta}_1$ (resp. $\tilde{\beta}_2$) is the tautological inclusion (resp. projection), and the diagram commutes that follows from the proof of [CL1, Prop 2.5 and 3.1].

Let \mathcal{N} be the coarse moduli space of the the stack $h^1/h^0((L_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}^\bullet)^\vee)|_{\tilde{\mathcal{Y}}_{\text{gst}}}$ relative to $\tilde{\mathcal{Y}}_{\text{gst}}$. Then $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}^\vee$ induces an embedding

$$\mathcal{N} \subset H^1(E_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}) = \tilde{V}_2.$$

Lemma 6.2. *Viewing \mathcal{N} as a substack of $\tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2$ via $\tilde{\beta}_1$ in (6.2), then*

$$(6.3) \quad \text{Supp}(C_{\text{gst}}) \cap (0 \oplus \tilde{V}_2) \subset \mathcal{N}.$$

Proof. It suffices to show that for any closed $y \in \tilde{\mathcal{Y}}_{\text{gst}}$, we have

$$(C_{\text{gst}} \cap (0 \oplus \tilde{V}_2))|_y \subset \mathcal{N}_y := \mathcal{N} \times_{\tilde{\mathcal{Y}}_{\text{gst}}} y.$$

First, it is shown in [BF] that $\mathcal{N}_y = H^1((L_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}^\bullet)^\vee|_y)$. Dualizing (6.1), restricting it to y and taking its cohomology groups, we obtain the commutative diagram of arrows

$$(6.4) \quad \begin{array}{ccccc} \mathcal{N}_y & \longrightarrow & H^1((L_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}^\bullet)^\vee|_y) & \longrightarrow & H^1(\gamma^*(L_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}}^\bullet)^\vee|_y) \\ \downarrow & & \downarrow \varrho_y & & \downarrow \\ \tilde{V}_2|_y & \xrightarrow{\tilde{\beta}_1|_y} & (\tilde{V}_1 \oplus \tilde{V}_2)|_y & \xrightarrow{\tilde{\beta}_2|_y} & \tilde{V}_1|_y, \end{array}$$

where the vertical arrows are H^1 of $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}^\vee|_y$, $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}^\vee|_y$ and $\gamma^*\phi_{\tilde{\mathcal{X}}/\tilde{\mathcal{D}}}^\vee|_y$, respectively, and the bottom line follows from (6.2). Since $\phi_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}$, etc., are perfect obstruction theories, the vertical arrows in (6.4) are injective.

We now prove (6.3). First, the inclusion $\mathbf{C}_{\text{gst}} \subset \mathbf{C}_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}} = h^1/h^0((L_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}^\bullet)^\vee)$ induces an inclusion of their respective fibers over y

$$\mathbf{C}_{\text{gst}}|_y \subset h^1/h^0((L_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}^\bullet)^\vee|_y) = h^1/h^0((L_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}^\bullet)^\vee|_y).$$

(Here we use the base change property of h^1/h^0 -construction; see [CL2, Lemm 2.3].) Taking coarse moduli, we have $C_{\text{gst}}|_y \subset H^1((L_{\tilde{\mathcal{Y}}/\tilde{\mathcal{D}}}^\bullet)^\vee|_y)$. Hence

$$(C_{\text{gst}} \cap (0 \oplus \tilde{V}_2))|_y = C_{\text{gst}}|_y \cap (0 \oplus \tilde{V}_2|_y) \subset \text{Im}(\varrho_y) \cap \ker(\tilde{\beta}_2|_y).$$

Because in (6.4) all vertical arrows are injective and the squares are commutative,

$$\text{Im}(\varrho_y) \cap \ker(\tilde{\beta}_2|_y) \subset 0 \oplus \mathcal{N}_y \subset 0 \oplus \tilde{V}_2|_y.$$

This proves (6.3). \square

Lemma 6.3. *The substack $\mathcal{N} \subset \tilde{V}_2$ is reduced and its support is the union of the zero section $0_{\tilde{V}_2} \subset \tilde{V}_2$ with the total space of a line subbundle $\tilde{\mathcal{F}} \subset \tilde{V}_2|_{\tilde{\Delta}}$ that is the pullback of a line subbundle $\mathcal{F} \subset V_2|_{\Delta}$.*

Proof. Let $y \in W = \tilde{\mathcal{Y}}_{\text{gst}}$ be a closed point. We pick a chart $\iota : \tilde{Y} \rightarrow \tilde{\mathcal{Y}}$ of $y \in \tilde{\mathcal{Y}}$ and its associated embedding given in Proposition 3.1. (We follow the notation introduced there.) Then by étale base change

$$\iota^* h^1/h^0((L_{\tilde{\mathcal{Y}}/\tilde{\mathcal{X}}}^\bullet)^\vee)|_{\tilde{Y}_{\text{gst}}} \cong h^1/h^0((L_{\tilde{Y}/\tilde{X}}^\bullet)^\vee)|_{\tilde{Y}_{\text{gst}}}.$$

We denote $Z'_{\tilde{X}} = \tilde{X} \times_Z Z' \cong \tilde{X} \times \mathbb{A}^1$. Lemma 3.1 induces a closed \tilde{X} -embedding $\tilde{Y} \rightarrow Z'_{\tilde{X}}$. Let I be the ideal of \tilde{Y} in $Z'_{\tilde{X}}$. By definition,

$$(6.5) \quad L_{\tilde{Y}/\tilde{X}}^{\bullet \geq -1} \cong [I/I^2 \xrightarrow{\delta} \Omega_{Z'_{\tilde{X}}/\tilde{X}}|_{\tilde{Y}}],$$

where $\delta(a) = d_{\tilde{X}}a$. ($d_{\tilde{X}}$ is the relative differential that annihilates $\mathcal{O}_{\tilde{X}}$.) We denote $S^\bullet(\cdot)$ the symmetric product, and denote

$$N := \text{Spec } S_{\mathcal{O}_{\tilde{Y}_{\text{gst}}}}^\bullet(I/I^2 \otimes_{\mathcal{O}_{\tilde{Y}}} \mathcal{O}_{\tilde{Y}_{\text{gst}}}).$$

Following [BF], since $h^1/h^0((L_{\tilde{Y}/\tilde{X}}^\bullet)^\vee) = h^1/h^0((L_{\tilde{Y}/\tilde{X}}^{\bullet \geq -1})^\vee)$, we have

$$h^1/h^0((L_{\tilde{Y}/\tilde{X}}^\bullet)^\vee)|_{\tilde{Y}_{\text{gst}}} = [(\text{Spec } S_{\mathcal{O}_{\tilde{Y}}}^\bullet(I/I^2))/(T_{Z'_{\tilde{X}}/\tilde{X}}|_{\tilde{Y}})]|_{\tilde{Y}_{\text{gst}}} = [N/(T_{Z'_{\tilde{X}}/\tilde{X}}|_{\tilde{Y}_{\text{gst}}})].$$

Here the $T_{Z'_{\tilde{X}}/\tilde{X}}|_{\tilde{Y}_{\text{gst}}}$ -action on N is induced by the arrow δ in (6.5).

By Lemma 3.1, $I = (wt) \subset \mathcal{O}_{Z'_X} = \mathcal{O}_{\tilde{X}}[t]$. Using $\tilde{Y}_{\text{gst}} = (w = 0) \cap Z'_X$, we obtain $\delta|_{\tilde{Y}_{\text{gst}}}(wt) = d_{\tilde{X}}(wt)|_{\tilde{Y}_{\text{gst}}} = w \cdot dt|_{\tilde{Y}_{\text{gst}}} = 0$; hence the $T_{Z'_X/\tilde{X}}|_{\tilde{Y}_{\text{gst}}}$ -action on N is trivial, and the coarse moduli of $h^1/h^0((L_{\tilde{Y}/\tilde{X}})^\vee)|_{\tilde{Y}_{\text{gst}}}$ is N ; namely

$$\mathcal{N} \times_{\tilde{\mathcal{Y}}_{\text{gst}}} \tilde{Y}_{\text{gst}} = N.$$

We determine N . Define $\alpha : \mathcal{O}_{Z'} \rightarrow I/I^2 \otimes_{\mathcal{O}_{\tilde{Y}}} \mathcal{O}_{\tilde{Y}_{\text{gst}}}$ to be the homomorphism via $\alpha(1) = wt \otimes 1$; because $I = (wt)$, α is surjective. We observe $\alpha(w) = w^2 t \otimes 1 = wt \otimes w = 0$, because $w = 0 \in \mathcal{O}_{\tilde{Y}_{\text{gst}}}$; we also have $\alpha(z_i) = wz_i t \otimes 1 = 0$, because $wz_i = 0$ in $\mathcal{O}_{\tilde{X}} \subset I$. A further check shows that $\ker \alpha = (w, z_1, z_2, z_3, z_4)$; hence $I/I^2 \otimes_{\mathcal{O}_{\tilde{Y}}} \mathcal{O}_{\tilde{Y}_{\text{gst}}} \cong \mathcal{O}_{\tilde{Y}_\Delta}$, where $\tilde{Y}_\Delta = \tilde{Y} \times_{\tilde{\mathcal{Y}}} \tilde{\Delta}$. As $\text{Spec } S_{\mathcal{O}_{\tilde{Y}_{\text{gst}}}}^\bullet(\mathcal{O}_{\tilde{Y}_\Delta})$ is the union of \tilde{Y}_{gst} with a line bundle \tilde{F} over \tilde{Y}_Δ so that $\tilde{Y}_{\text{gst}} \cap \tilde{F} = \tilde{Y}_\Delta$, this proves that $N = \tilde{Y}_{\text{gst}} \cup \tilde{F}$ with $\tilde{Y}_{\text{gst}} \cap \tilde{F} = \tilde{Y}_\Delta$.

As $\tilde{Y} \rightarrow \tilde{\mathcal{Y}}$ is an arbitrary étale chart, it shows that there is a line bundle $\tilde{\mathcal{F}}$ on $\tilde{\Delta} \subset \tilde{\mathcal{Y}}_{\text{gst}}$ such that

- (1) both $\tilde{\mathcal{Y}}_{\text{gst}}$ and $\tilde{\mathcal{F}}$ are closed substacks of \mathcal{N} ;
- (2) $\mathcal{N} = \tilde{\mathcal{Y}}_{\text{gst}} \cup \tilde{\mathcal{F}}$ and $\tilde{\mathcal{Y}}_{\text{gst}} \cap \tilde{\mathcal{F}} = \tilde{\Delta}$, where $\tilde{\mathcal{Y}}_{\text{gst}} \subset \mathcal{N}$ is the zero section $0_{\tilde{V}_2}$.

Since $\mathcal{N} \subset \tilde{V}_2$ is a subcone, $\tilde{\mathcal{F}} = \mathcal{N} \times_{\tilde{\mathcal{Y}}_{\text{gst}}} \tilde{\Delta} \subset \tilde{V}_2|_{\tilde{\Delta}}$ is also a subcone and we conclude that $\tilde{\mathcal{F}} \subset \tilde{V}_2|_{\tilde{\Delta}}$ is a subline bundle.

Finally, we show that there is a subline bundle $\mathcal{F} \subset V_2|_{\Delta}$ that pullbacks to $\tilde{\mathcal{F}} \subset \tilde{V}_2|_{\tilde{\Delta}}$. For this, we use the \mathbb{C}^* -action on $\tilde{\mathcal{Y}}_{\text{gst}}$ introduced in the proof of Lemma 5.7. By Lemma 5.7, $\tilde{\mathcal{Y}}_{\text{gst}} - \tilde{\mathcal{X}}_{\text{gst}} \rightarrow \tilde{\mathcal{X}}_{\text{gst}}$ is a \mathbb{C}^* -quotient morphism. By its construction, the relative obstruction theory of $\tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{X}}$ is \mathbb{C}^* -equivariant. Thus $\tilde{\mathcal{F}}|_{\tilde{\mathcal{Y}}_{\text{gst}} - \tilde{\mathcal{X}}_{\text{gst}}} \subset \tilde{V}_2|_{\tilde{\mathcal{Y}}_{\text{gst}} - \tilde{\mathcal{X}}_{\text{gst}}}$ is \mathbb{C}^* -equivariant, where the \mathbb{C}^* -action on $\tilde{V}_2|_{\tilde{\mathcal{Y}}_{\text{gst}} - \tilde{\mathcal{X}}_{\text{gst}}}$ is via the linearization $\Phi_{2,1}$.

As this linearization $\Phi_{2,1}$ differs from $\Phi_{2,0}$ by a scalar multiplication, and because $\tilde{\mathcal{F}}|_y \subset \tilde{V}_2|_y$ for $y \in \tilde{\Delta}$ is a linear subspace, $\tilde{\mathcal{F}}|_{\tilde{\Delta}} \subset \tilde{V}_2|_{\tilde{\Delta}}$ is also invariant under the linearization $\Phi_{2,0}$. Since $\Phi_{2,0}$ is induced from the pullback $\tilde{V}_2 = \gamma^* V_2$, by descent theory, $\tilde{\mathcal{F}}|_{\tilde{\Delta} - \Delta} \subset \tilde{V}_2|_{\tilde{\Delta} - \Delta}$ descends to a subline bundle $\mathcal{F} \subset V_2|_{\Delta}$. (As usual, we view $\Delta \subset \tilde{\Delta}$ via the zero section of $W = \text{Total}(L)$.) Thus $\tilde{\mathcal{F}}|_{\tilde{\Delta} - \Delta} = \gamma^* \mathcal{F}|_{\tilde{\Delta} - \Delta} \subset \tilde{V}_2|_{\tilde{\Delta} - \Delta}$. Finally, since $\tilde{\Delta} - \Delta$ is dense in $\tilde{\Delta}$, we have $\tilde{\mathcal{F}} = \gamma^* \mathcal{F} \subset \tilde{V}_2|_{\tilde{\Delta}}$. This proves the Lemma. \square

We now apply Corollary 5.2 to the situation $W = \tilde{\mathcal{Y}}_{\text{gst}}$ and $M = \tilde{\mathcal{X}}_{\text{gst}}$. As will be clear later, we will work with each individual component $W_\mu = \tilde{\mathcal{Y}}_\mu$ of $\tilde{\mathcal{Y}}_{\text{gst}}$, and the corresponding components $M_\mu = \tilde{\mathcal{X}}_\mu$. Accordingly, we will add the subscript μ to denote the corresponding objects restricting to M_μ , W_μ or $\overline{W}_\mu = \overline{W} \times_M M_\mu$. For instance, $\overline{V}_{i,\mu} = \overline{V}_i|_{\overline{W}_\mu}$, $\Delta_\mu = \Delta \cap M_\mu$ and $\tilde{\Delta}_\mu = \tilde{\Delta} \times_M M_\mu$, etc.

Following (3.3), the cycle $[C_{\text{gst}}] \in Z_* \tilde{V}$ is a union $[C_{\text{gst}}] = \sum_{\mu^d} \iota_{\mu*} [C_\mu]$, where $[C_\mu] \in Z_* \tilde{V}_\mu$ and $\iota_\mu : \tilde{V}_\mu \rightarrow \tilde{V}$ is the inclusion. Here as argued before (3.2), $[C_\mu] = \sum_k n_{\mu,k} [C_{\mu,k}]$ could be a union of integral multiples of irreducible components $C_{\mu,k}$. Let \overline{C}_μ be the closure of C_μ in \overline{V}_μ , which by the given irreducible decomposition is $\sum_k n_{\mu,k} [\overline{C}_{\mu,k}]$; we let $\overline{C}_{\mu,k,b} = \overline{C}_{\mu,k} \cap (0 \oplus \overline{V}_{2,\mu})$, and let

$$R_\mu = \sum_k n_{\mu,k} N_{\overline{C}_{\mu,k,b}} \overline{C}_{\mu,k}$$

be the normal cone to $\overline{C}_{\mu,b} = \sum_k n_{\mu,k} \overline{C}_{\mu,k,b}$ in \overline{C}_μ .

By Corollary 5.2, we have

$$(6.6) \quad \deg 0_{\xi, \text{loc}}^! [C_{\text{gst}}] = \deg 0_{\overline{V}}^! [\overline{C}_{\text{gst}}] = \sum_{\mu \vdash d} \deg 0_{\overline{V}_\mu}^! [R_\mu].$$

We look more closely the term $0_{\overline{V}_\mu}^! [R_\mu]$. By Lemma 6.3 we have a line subbundle $\mathcal{F}_\mu := \mathcal{F}|_{\Delta_\mu} \subset V_{2,\mu}|_{\Delta_\mu} = V_2|_{\Delta_\mu}$. By Lemma 5.4, Lemma 5.7 and Proposition 6.1, $[\overline{C}_{\mu,b}]$ lies over the union of the zero section $0_{\overline{V}_\mu}$ with the total space $\overline{F}_\mu = \text{Total}(\overline{\mathcal{F}}_\mu)$, where $\overline{\Delta}_\mu = \overline{W}_\mu \times_{M_\mu} \Delta_\mu$, and $\bar{\gamma}_\mu := \bar{\gamma}|_{\overline{W}_\mu} : \overline{W}_\mu \rightarrow M_\mu$. We let

$$(6.7) \quad Z_\mu = 0_{\overline{V}_{2,\mu}} \cup \overline{F}_\mu.$$

Thus R_μ lies in the bundle $\overline{V}_{1,\mu} \times_{\overline{W}_\mu} Z_\mu$ over Z_μ , where $Z_\mu \rightarrow \overline{W}_\mu$ is the identity map when restricted to $0_{\overline{V}_{2,\mu}}$, and is composite of the projection $\overline{F}_\mu \rightarrow \overline{\Delta}_\mu \rightarrow \overline{W}_\mu$.

We claim that $\dim \overline{W}_\mu = 5d+4$, $\dim R_\mu = 5d+6$, $\text{rank } \overline{V}_{1,\mu} = 5$ and $\text{rank } \overline{V}_{2,\mu} = 5d+1$. Indeed, by Proposition 3.1, we know that all $\tilde{\mathcal{X}}_\mu$ are smooth and are of equal dimensions. Thus

$$\dim \tilde{\mathcal{X}}_\mu = \dim \tilde{\mathcal{X}}_{(d)} = \dim \overline{M}_{0,1}(\mathbb{P}^4, d) + \dim \overline{M}_{1,1} = 5d+3.$$

As $\tilde{\mathcal{Y}}_\mu$ is a line bundle over $\tilde{\mathcal{X}}_\mu$, we obtain $\dim \overline{W}_\mu = \dim \tilde{\mathcal{Y}}_\mu = 5d+4$. For $\dim R_\mu$, since C_μ has pure-dimension, and $\dim R_\mu = \dim C_\mu$, we only need to verify $\dim C_\mu = \dim \tilde{\mathcal{Y}}_\mu + 2$. But this follows from (2) of Lemma 3.3. The remainder two identities follows from Riamenn-Roch theorem.

We denote by $|\overline{C}_{\mu,b}|$ the support of $\overline{C}_{\mu,b}$, which is the union $\cup_k \overline{C}_{\mu,k,b}$. Since R_μ is a normal cone, by its construction, it is a cone inside the vector bundle $\overline{V}_{1,\mu} \times_{\overline{W}_\mu} |\overline{C}_{\mu,b}| \subset \overline{V}_{1,\mu} \oplus \overline{V}_{2,\mu}$. (Here as total space of vector bundles, $\overline{V}_{1,\mu} \times_{\overline{W}_\mu} \overline{V}_{2,\mu} = \overline{V}_{1,\mu} \oplus \overline{V}_{2,\mu}$.)

Therefore, we have

$$(6.8) \quad 0_{\overline{V}_{1,\mu}}^! [R_\mu] \in A_{5d+1} |\overline{C}_{\mu,b}|.$$

We write $0_{\overline{V}_{1,\mu}}^! [R_\mu] = [P_{\mu,1}] + [P_{\mu,2}]$, where $P_{\mu,1} \subset 0_{\overline{V}_\mu}$ and $P_{\mu,2} \subset \overline{F}_\mu$.

Lemma 6.4. *We have the vanishing $\deg 0_{\overline{V}_{2,\mu}}^! [P_{\mu,i}] = 0$ in the following cases:*

- (1) $\mu \vdash d$ and $i = 2$;
- (2) $\mu \neq (d) \vdash d$ and $i = 1$.

Proof. We prove the vanishing for $i = 2$. By Lemma 6.3, the subline bundle $\mathcal{F}_\mu \subset V_1|_{\Delta_\mu}$ pullbacks to $\overline{\mathcal{F}}_\mu \subset \overline{V}_{2,\mu}|_{\overline{\Delta}_\mu}$. Let $\eta_\mu : \overline{\mathcal{F}}_\mu \rightarrow \mathcal{F}_\mu$ be the projection; it is proper since $\overline{\Delta} \rightarrow \Delta$ is proper. Then by the projection formula,

$$\deg 0_{\overline{V}_{2,\mu}}^! [P_{\mu,2}] = \deg 0_{V_{2,\mu}}^! (\eta_{\mu*} [P_{\mu,2}]).$$

Since $\eta_{\mu*} [P_{\mu,2}] \in A_{5d+1} \mathcal{F}_\mu$ and $\dim \mathcal{F}_\mu = \dim \Delta_\mu + 1 = (\dim \tilde{\mathcal{X}}_{\text{pri}} - 1) + 1 = 5d$, we have $\eta_{\mu*} [P_{\mu,2}] = 0$. This proves the first vanishing.

To prove the second vanishing, we will construct a proper DM stack $\tilde{\mathcal{B}}_\mu$, a vector bundle $\tilde{\mathcal{X}}_\mu$ on $\tilde{\mathcal{B}}_\mu$, and a proper morphism with an isomorphism

$$(6.9) \quad \rho_\mu : M_\mu = \tilde{\mathcal{X}}_\mu \longrightarrow \tilde{\mathcal{B}}_\mu \quad \text{and} \quad \rho_\mu^* \tilde{\mathcal{X}}_\mu^\vee \cong V_{2,\mu},$$

such that $\dim \tilde{\mathcal{B}}_{(d)} = 5d + 1$, and $\dim \tilde{\mathcal{B}}_\mu \leq 5d$ for $\mu \neq (d)$. Once $(\tilde{\mathcal{B}}_\mu, \tilde{\mathcal{K}}_\mu)$ is constructed, we let $\delta_\mu : \overline{V}_{2,\mu} \rightarrow \tilde{\mathcal{K}}_\mu^\vee$ be the projection induced by the mentioned isomorphism; then, for $\mu \neq (d)$, by the projection formula we have $\deg 0_{\overline{V}_{2,\mu}}^! [P_{\mu,1}] = \deg 0_{\tilde{\mathcal{K}}_\mu^\vee}^! (\delta_{\mu*} [P_{\mu,1}]) = 0$ because $P_{\mu,1} \subset 0_{\overline{V}_{2,\mu}}$ and $A_{5d+1} \tilde{\mathcal{B}}_\mu = 0$ for dimension reason.

Constructing $\tilde{\mathcal{B}}_\mu$ and $\tilde{\mathcal{K}}_\mu$ with the required properties will occupy the remainder of this Section. \square

We first state a decomposition result, which follows from the construction in [HL1]. Let $\mu = (d_1, \dots, d_\ell)$ be a partition of d and let $(f_{\tilde{\mathcal{X}}_\mu}, \pi_{\tilde{\mathcal{X}}_\mu}, \mathcal{C}_{\tilde{\mathcal{X}}_\mu})$ be the tautological family of $\tilde{\mathcal{X}}_\mu$. By the construction of $\tilde{\mathcal{X}}_{\text{gst}}$, the map associated to a closed point in $\tilde{\mathcal{X}}_\mu$ is by attaching ℓ one-pointed $[u_i, C_i, p_i] \in \overline{M}_{0,1}(\mathbb{P}^4, d_i)$ such that $u_i(p_i) = u_j(p_j)$ for all i, j to an ℓ -pointed stable elliptic curve. We state this in the family version

Proposition 6.5. *The tautological family $\mathcal{C}_{\tilde{\mathcal{X}}_\mu} \rightarrow \tilde{\mathcal{X}}_\mu$ admits an ℓ -section $\Sigma_\mu \subset \mathcal{C}_{\tilde{\mathcal{X}}_\mu}$ (a codimension one closed substack, proper, and an ℓ -étale cover of $\tilde{\mathcal{X}}_\mu$) that lies in the locus of nodal points of the fibers of $\mathcal{C}_{\tilde{\mathcal{X}}_\mu}/\tilde{\mathcal{X}}_\mu$, and splits $\mathcal{C}_{\tilde{\mathcal{X}}_\mu}$ into two families of curves: $\mathcal{C}_{\tilde{\mathcal{X}}_\mu, \text{pr}}$ and $\mathcal{C}_{\tilde{\mathcal{X}}_\mu, \text{tl}}$ ($\subset \mathcal{C}_{\tilde{\mathcal{X}}_\mu}$), such that*

- (1) *the pair $(\mathcal{C}_{\tilde{\mathcal{X}}_\mu, \text{pr}}, \Sigma_\mu)$ is a family of ℓ -pointed (unordered) stable genus one curves; the morphism $f_{\tilde{\mathcal{X}}_\mu}$ is constant along fibers of $\mathcal{C}_{\tilde{\mathcal{X}}_\mu, \text{pr}} \rightarrow \tilde{\mathcal{X}}_\mu$;*
- (2) *the pair $(\mathcal{C}_{\tilde{\mathcal{X}}_\mu, \text{tl}}, \Sigma_\mu)$ is a family of ℓ -pointed (unordered) nodal rational curves over $\tilde{\mathcal{X}}_\mu$ such that each closed fiber of $\mathcal{C}_{\tilde{\mathcal{X}}_\mu, \text{tl}} \rightarrow \tilde{\mathcal{X}}_\mu$ has ℓ connected components, and each connected component contains one marked point.*

Here the subscript “pr” stands for the “principal part” and the subscript “tl” stands for the “tail”. We comment that the total space $\overline{\mathcal{C}}_{\tilde{\mathcal{X}}_\mu, \text{tl}}$ may have less than ℓ connected components.

Proof. The proof follows from the modular construction of $\tilde{\mathcal{X}}_\mu$ in [HL1]. \square

We now fix a partition μ and assume $\ell \geq 2$. Using this decomposition, we can relate $\tilde{\mathcal{X}}_\mu$ to a stack that parameterizes the tails of $[u, C] \in \tilde{\mathcal{X}}_\mu$. We take the moduli of genus zero stable morphisms $\overline{M}_{0,1}(\mathbb{P}^4, d_i)$, considered as a stack over \mathbb{P}^4 via the evaluation morphism $\text{ev}_i : \overline{M}_{0,1}(\mathbb{P}^4, d_i) \rightarrow \mathbb{P}^4$ (of the marked points); we form

$$\mathcal{B}_\mu = \overline{M}_{0,1}(\mathbb{P}^4, d_1) \times_{\mathbb{P}^4} \cdots \times_{\mathbb{P}^4} \overline{M}_{0,1}(\mathbb{P}^4, d_\ell).$$

We let S_μ be the subgroup of permutations $\alpha \in S_\ell$ that leave the ℓ -tuple (d_1, \dots, d_ℓ) invariant (i.e. $d_i = d_{\alpha(i)}$ for all $1 \leq i \leq \ell$). Each $\alpha \in S_\mu$ acts as an automorphism of \mathcal{B}_μ by permuting the i -th and the $\alpha(i)$ -th factors of \mathcal{B}_μ . This gives an S_μ -action on \mathcal{B}_μ ; we define the stacky quotient

$$\tilde{\mathcal{B}}_\mu = \mathcal{B}_\mu / S_\mu.$$

Since $\overline{M}_{0,1}(\mathbb{P}^4, d_i)$ are proper DM stacks and have dimensions $5d_i + 2$, $\tilde{\mathcal{B}}$ is a proper DM-stack and

$$(6.10) \quad \dim \tilde{\mathcal{B}}_\mu = (5d + 2\ell) - (4\ell - 4) = 5d - 2\ell + 4.$$

For each i we denote the universal family of $\overline{M}_{0,1}(\mathbb{P}^4, d_i)$ by $(\pi_i, f_i) : \mathcal{C}_i \rightarrow \overline{M}_{0,1}(\mathbb{P}^4, d_i) \times \mathbb{P}^4$ with $s_i : \overline{M}_{0,1}(\mathbb{P}^4, d_i) \rightarrow \mathcal{C}_i$ the section of marked point. We introduce

$$\mathcal{K}_i = \pi_{i*} f_i^* \mathcal{O}(5),$$

a rank $5d_i + 1$ locally free sheaf (vector bundle) on $\overline{M}_{0,1}(\mathbb{P}^4, d_i)$ with an evaluation homomorphism $e_i : \mathcal{K}_i \rightarrow \text{ev}_i^* \mathcal{O}(5)$.

We let $v_i : \mathcal{B}_\mu \rightarrow \overline{M}_{0,1}(\mathbb{P}^4, d_i)$ be the projection. Since \mathcal{B}_μ is constructed as the fiber-product using the evaluations ev_i , the collection $\{\text{ev}_i\}_{i=1}^\ell$ descends to a single evaluation morphism $\text{ev} : \mathcal{B}_\mu \rightarrow \mathbb{P}^4$. We form a sheaf \mathcal{K}_μ on \mathcal{B}_μ defined via the exact sequence

$$0 \longrightarrow \mathcal{K}_\mu \longrightarrow \bigoplus_{i=1}^\ell v_i^* \mathcal{K}_i \xrightarrow{\beta} \bigoplus_{j=1}^{\ell-1} \text{ev}^* \mathcal{O}(5) \longrightarrow 0,$$

where $\beta = (v_1^* e_1 - v_2^* e_2, \dots, v_{\ell-1}^* e_{\ell-1} - v_\ell^* e_\ell)$. Since each $e_i : \mathcal{K}_i \rightarrow \text{ev}_i^* \mathcal{O}(5)$ is surjective, \mathcal{K}_μ is locally free. Further, \mathcal{K}_μ is invariant under S_μ , thus descends to a locally free sheaf $\tilde{\mathcal{K}}_\mu$ on $\tilde{\mathcal{B}}_\mu$. (Caution, the arrow β is not invariant under \sim , due to the choice of indexing.) As $\text{rank } \mathcal{K}_i = 5d_i + 1$, $\text{rank } \tilde{\mathcal{K}}_\mu = 5d + 1$.

We define the desired morphism $\rho_\mu : \tilde{\mathcal{X}}_\mu \rightarrow \tilde{\mathcal{B}}_\mu$ stated in (6.9). Given any closed $x \in \tilde{\mathcal{X}}_\mu$, we let $[f_x, \mathcal{C}_x]$ with $\Sigma_x \subset \mathcal{C}_x$ be the restriction of $f_{\tilde{\mathcal{X}}_\mu}$ and Σ_μ to $\mathcal{C}_x = \mathcal{C}_{\tilde{\mathcal{X}}_\mu} \times_{\tilde{\mathcal{X}}_\mu} x$. By Proposition 6.5, Σ_x divides \mathcal{C}_x into a union of a genus one curve with ℓ one-pointed genus zero curves, and the restriction of f_x to these ℓ genus zero curves form ℓ one-pointed genus zero stable maps to \mathbb{P}^4 , of degrees d_1, \dots, d_ℓ , respectively. We label these ℓ -stable maps as $[u_i, C_i, p_i]$ so that $[u_i] \in \overline{M}_{0,1}(\mathbb{P}^4, d_i)$. We define $\rho_\mu(x)$ be the equivalence class in $\tilde{\mathcal{B}}_\mu$ of $([u_1], \dots, [u_\ell]) \in \mathcal{B}_\mu$. It is routine to check that this pointwise definition defines a morphism ρ_μ as indicated.

We verify that $\rho_\mu^* \tilde{\mathcal{K}}_\mu^\vee \cong V_{2,\mu}$. First, by Serre duality,

$$V_{2,\mu} = R^1 \pi_{\tilde{\mathcal{X}}_\mu *} \mathcal{P}_{\tilde{\mathcal{X}}_\mu} \cong (\pi_{\tilde{\mathcal{X}}_\mu *} f_{\tilde{\mathcal{X}}_\mu}^* \mathcal{O}(5))^\vee.$$

Let $f_{\tilde{\mathcal{X}}_\mu, \text{tl}}$ and $\pi_{\tilde{\mathcal{X}}_\mu, \text{tl}}$ be the restrictions of $f_{\tilde{\mathcal{X}}_\mu}$ and $\pi_{\tilde{\mathcal{X}}_\mu}$ to $\mathcal{C}_{\tilde{\mathcal{X}}_\mu, \text{tl}}$. We obtain the restriction homomorphism

$$(6.11) \quad \pi_{\tilde{\mathcal{X}}_\mu *} f_{\tilde{\mathcal{X}}_\mu} \mathcal{O}(5) \xrightarrow{\subset} (\pi_{\tilde{\mathcal{X}}_\mu, \text{tl}})_* f_{\tilde{\mathcal{X}}_\mu, \text{tl}}^* \mathcal{O}(5).$$

Since fibers of $\mathcal{C}_{\tilde{\mathcal{X}}_\mu, \text{pr}} \rightarrow \tilde{\mathcal{X}}_\mu$ are connected, and $f_{\tilde{\mathcal{X}}_\mu}$ restricting to them are constants, (6.11) is injective and its cokernel is the difference of the evaluations along $\Sigma_\mu \subset \mathcal{C}_{\tilde{\mathcal{X}}_\mu, \text{tl}}$. On the other hand, denoting $(\bigoplus_{i=1}^\ell v_i^* \mathcal{K}_i) / S_\mu$ the quotient of $\bigoplus_{i=1}^\ell v_i^* \mathcal{K}_i$ over \mathcal{B}_μ by S_μ , (which is its descent to $\tilde{\mathcal{B}}_\mu$), then via ρ_μ , we have a canonical isomorphism

$$\rho_\mu^* ((\bigoplus_{i=1}^\ell v_i^* \mathcal{K}_i) / S_\mu) \cong (\pi_{\tilde{\mathcal{X}}_\mu, \text{tl}})_* f_{\tilde{\mathcal{X}}_\mu, \text{tl}}^* \mathcal{O}(5).$$

A direct inspection shows that under this isomorphism, $\rho_\mu^* \tilde{\mathcal{K}}_\mu \cong \pi_{\tilde{\mathcal{X}}_\mu *} f_{\tilde{\mathcal{X}}_\mu}^* \mathcal{O}(5)$. This proves $\rho_\mu^* \tilde{\mathcal{K}}_\mu^\vee \cong V_{2,\mu}$.

In case $\mu = (d)$, we let $\tilde{\mathcal{B}}_{(d)} = \overline{M}_0(\mathbb{P}^4, d)$, and all others are the same.

Completing the proof of Lemma 6.4. The pair $(\tilde{\mathcal{B}}_\mu, \mathcal{K}_\mu)$ satisfies the requirements stated in the proof of Lemma 6.4, except possibly the dimensions part. By (6.10), for $\ell \geq 2$, we have $\dim \tilde{\mathcal{B}}_\mu \leq 5d$; for $\mu = (d)$, we have $\dim \tilde{\mathcal{B}}_{(d)} = \dim \overline{M}_0(\mathbb{P}^4, d) = 5d + 1$, which are as required. This completes the proof of Lemma 6.4. \square

7. PROOF OF PROPOSITION 3.6

In this section, we treat the remainder case left out in Lemma 6.4. Following the discussion in the previous section, we know that $[P_{(d),1}] \in A_{5d+1} \overline{W}_{(d)}$ and $\tilde{\mathcal{B}}_{(d)}$ is irreducible and of dimension $5d+1$; thus

$$(7.1) \quad (\rho_{(d)} \circ \bar{\gamma}_{(d)})_* [P_{(d),1}] = c[\tilde{\mathcal{B}}_{(d)}]$$

for a $c \in \mathbb{Q}$. Applying the projection formula to $\rho_{(d)} \circ \bar{\gamma}_{(d)}$, we obtain

$$(7.2) \quad \deg 0_{\overline{V}_{2,(d)}}^1 [P_{(d),1}] = c \cdot \deg c_{5d+1}(\tilde{\mathcal{K}}_{(d)})[\tilde{\mathcal{B}}_{(d)}] = c \cdot (-1)^{5d+1} N_0(d)_Q.$$

Therefore, all we need is to determine c .

To this end, we need to determine the cone $C_{(d)} \subset \tilde{V}|_{W_{(d)}}$. By the construction in [HL1], $\overline{M}_1(\mathbb{P}^4, d)$ contains an open substack consisting of stable maps $[u, C]$ such that their domains C are union of elliptic curves with \mathbb{P}^1 , and the maps u restricted to the elliptic curves are constant and restricted to \mathbb{P}^1 are regular embeddings $\mathbb{P}^1 \rightarrow \mathbb{P}^4$. We denote this open substack by $M_o \subset \overline{M}_1(\mathbb{P}^4, d)$. By [HL1], M_o is away from the blowing up loci of the modular blowing up $\tilde{\mathcal{X}} \rightarrow \mathcal{X} = \overline{M}_1(\mathbb{P}^4, d)$. Thus, the preimage of M_o in $\tilde{\mathcal{X}}$ is identical to M_o , and thus $\tilde{\mathcal{Y}} \times_{\mathcal{X}} M_o = \mathcal{Y} \times_{\mathcal{X}} M_o$.

We denote $W_o = \mathcal{Y} \times_{\mathcal{X}} M_o$. By the prior discussion, we know $W_o = \tilde{\mathcal{Y}} \times_{\mathcal{X}} M_o \subset W_{(d)}$, and is dense in $W_{(d)}$. Since we need to work with various obstruction theories that are originally constructed for \mathcal{Y} and \mathcal{X} , we will view $M_o \subset M_{(d)}$ and also an open substack of \mathcal{X} ; same for $W_o \subset W_{(d)}$ and $W_o \subset \mathcal{Y}$.

Since $W_o \subset \mathcal{Y}$ is open, the obstruction theory $\phi_{\mathcal{Y}/\mathcal{D}}$ restricted to W_o gives a perfect relative obstruction theory $\phi_{\mathcal{Y}/\mathcal{D}}|_{W_o} : (E_{\mathcal{Y}/\mathcal{D}})^\vee|_{W_o} \rightarrow L_{W_o/\mathcal{D}}^\bullet$.

Lemma 7.1. *Restricting to W_o , we have*

$$C_{(d)}|_{W_o} = H^1((L_{W_o/\mathcal{D}}^\bullet)^\vee) \xrightarrow{\subseteq} H^1(E_{\mathcal{Y}/\mathcal{D}}|_{W_o}) = \tilde{V}|_{W_o}.$$

Further, $H^1((L_{W_o/\mathcal{D}}^\bullet)^\vee)$ is a rank two locally free sheaf of \mathcal{O}_{W_o} -modules, and the arrow above is injective with locally free cokernel.

Proof. Let $q_o : W_o \rightarrow \mathcal{D}$ be the projection induced by $\mathcal{Y} \rightarrow \mathcal{D}$; let $\mathcal{D}_o = q_o(W_o)$ be its image stack. By the description of W_o and the argument in the proof of Lemma 3.3, $\mathcal{D}_o \subset \mathcal{D}$ is a smooth codimension two locally closed substack. By [Il, Chap. III Prop. 3.2.4], $H^1(L_{\mathcal{D}_o/\mathcal{D}}^\vee)$ is isomorphic to the normal sheaf to \mathcal{D}_o in \mathcal{D} , which is a rank two locally free sheaf on \mathcal{D}_o .

On the other hand, since $W_o \rightarrow \mathcal{D}_o$ is smooth, [Il, Chap. III Prop. 3.1.2] implies that $H^i((L_{W_o/\mathcal{D}_o}^\bullet)^\vee) = 0$ for $i \geq 1$. Taking H^1 of the distinguished triangle

$$(L_{W_o/\mathcal{D}_o}^\bullet)^\vee \longrightarrow (L_{W_o/\mathcal{D}}^\bullet)^\vee \longrightarrow q_o^*(L_{\mathcal{D}_o/\mathcal{D}}^\bullet)^\vee \xrightarrow{+1},$$

we obtain canonical

$$q_o^* H^1((L_{\mathcal{D}_o/\mathcal{D}}^\bullet)^\vee) \cong H^1((L_{W_o/\mathcal{D}}^\bullet)^\vee),$$

thus they are rank two locally free sheaves on W_o .

Next, for any closed point $y \in \mathcal{Y}$, since $\phi_{\mathcal{Y}/\mathcal{D}}$ is a perfect obstruction theory, $H^1(\phi_{\mathcal{Y}/\mathcal{D}}^\vee|_y)$ is injective. This combined with W_o being smooth shows that the arrow in the statement of Lemma 7.1 is injective with locally free cokernel.

Finally, we show that $C_{(d)}|_{W_o} = H^1((L_{W_o/\mathcal{D}}^\bullet)^\vee)$. Because $W_o \rightarrow \mathcal{D}_o$ is smooth and $\mathcal{D}_o \subset \mathcal{D}$ is smooth of codimension two, we conclude that $L_{W_o/\mathcal{D}}^\bullet$ is perfect of amplitude $[0, 1]$, and $H^i((L_{W_o/\mathcal{D}}^\bullet)^\vee)$ are locally free. Hence the coarse

moduli of $h^1/h^0((L_{W_\circ/\mathcal{D}}^\bullet)^\vee)$ is $H^1((L_{W_\circ/\mathcal{D}}^\bullet)^\vee)$. By [BF, Prop 3.1.2], $\mathbf{C}_{\mathcal{Y}/\mathcal{D}}|_{W_\circ} = h^1/h^0((L_{W_\circ/\mathcal{D}}^\bullet)^\vee)$. Taking coarse moduli, we obtain $C_{(d)}|_{W_\circ} = H^1(L_{W_\circ/\mathcal{D}}^\bullet)^\vee$. This proves the Lemma. \square

Let $\gamma_\circ : W_\circ \rightarrow M_\circ$ be the projection induced by $\mathcal{Y} \rightarrow \mathcal{X}$. We denote by \mathcal{M} the Artin stack of genus one nodal curves; (this is consistent with \mathcal{M}^w is the stack of weighted genus one nodal curves). Since $\mathcal{M}^w \rightarrow \mathcal{M}$ is étale, the obstruction theory of $\mathcal{X} \rightarrow \mathcal{M}$ is the same as that of $\mathcal{X} \rightarrow \mathcal{M}^w$.

We determine the subsheaf $H^1((L_{W_\circ/\mathcal{D}}^\bullet)^\vee) \subset \tilde{V}|_{W_\circ}$ by studying the following diagrams:

$$(7.3) \quad \begin{array}{ccccc} H^1((L_{W_\circ/\mathcal{D}}^\bullet)^\vee) & \xrightarrow{\alpha_1} & \gamma_\circ^* H^1((L_{M_\circ/\mathcal{D}}^\bullet)^\vee) & \xrightarrow{\alpha_2} & \gamma_\circ^* H^1((L_{M_\circ/\mathcal{M}}^\bullet)^\vee) \\ \downarrow H^1(\phi_{\mathcal{Y}/\mathcal{D}}^\vee) & & \downarrow H^1(\phi_{\mathcal{X}/\mathcal{D}}^\vee) & & \downarrow H^1(\phi_{\mathcal{X}/\mathcal{M}}^\vee) \\ H^1(E_{\mathcal{Y}/\mathcal{D}}|_{W_\circ}) & \xrightarrow{\tilde{\beta}_2|_{W_\circ}} & \gamma_\circ^* H^1(E_{\mathcal{X}/\mathcal{D}}|_{M_\circ}) & \xrightarrow{\gamma_\circ^* \beta_\circ} & \gamma_\circ^* H^1(E_{\mathcal{X}/\mathcal{M}}|_{M_\circ}). \end{array}$$

Here, $\tilde{\beta}_i$ is defined in (6.2) and β_\circ is the tautological projection induced by the comparison of the obstruction theories of $\mathcal{X} \rightarrow \mathcal{D}$ and $\mathcal{X} \rightarrow \mathcal{M}$:

$$(7.4) \quad \beta_\circ : H^1(E_{\mathcal{X}/\mathcal{D}}|_{M_\circ}) \longrightarrow H^1(E_{\mathcal{X}/\mathcal{M}}|_{M_\circ}).$$

We comment that the left commutative square is induced by that the obstruction theories (relative to \mathcal{D}) of $\mathcal{Y} \subset C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P}))$ and of $\mathcal{X} \subset C(\pi_*(\mathcal{L}^{\oplus 5}))$ are compatible under $C(\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{P})) \rightarrow C(\pi_*(\mathcal{L}^{\oplus 5}))$ induced by the projection $\text{pr} : \mathcal{L}^{\oplus 5} \oplus \mathcal{P} \rightarrow \mathcal{L}^{\oplus 5}$. The right square is commutative following [CL1, Lemm 2.8].

We let $(f_{M_\circ}, \pi_{M_\circ}) : \mathcal{C}_{M_\circ} \rightarrow \mathbb{P}^4 \times M_\circ$ be the universal family of $M_\circ \subset \mathcal{X}$.

Lemma 7.2. *All sheaves in the diagram (7.3) are locally free sheaves of \mathcal{O}_{W_\circ} -modules; all vertical arrows are injective with locally free cokernels; the arrow α_1 is an isomorphism; the arrow α_2 is surjective and has rank one kernel; the arrow $\tilde{\beta}_2|_{W_\circ}$ is the obvious projection $\tilde{V}|_{W_\circ} = (\tilde{V}_1 \oplus \tilde{V}_2)|_{W_\circ} \rightarrow \tilde{V}_1|_{W_\circ}$; the arrow β_\circ is the projection*

$$\beta_\circ : H^1(E_{\mathcal{X}/\mathcal{D}}|_{M_\circ}) = R^1\pi_{M_\circ*}f_{M_\circ}^*\mathcal{O}(1)^{\oplus 5} \longrightarrow H^1(E_{\mathcal{X}/\mathcal{M}}|_{M_\circ}) = R^1\pi_{M_\circ*}f_{M_\circ}^*T_{\mathbb{P}^4}$$

induced by the tautological projection $\mathcal{O}(1)^{\oplus 5} \rightarrow T_{\mathbb{P}^4}$ in the Euler sequence of \mathbb{P}^4 . Finally, the cokernels of the middle and the third vertical arrows are isomorphic.

Proof. We let $\mathcal{M}_\circ \subset \mathcal{M}$ be the image stack of $M_\circ \rightarrow \mathcal{M}$. By the description of M_\circ , \mathcal{M}_\circ is a locally closed smooth divisor in \mathcal{M} . Thus the normal sheaf $\mathcal{N}_{\mathcal{M}_\circ/\mathcal{M}}$ is invertible and the canonical

$$(7.5) \quad \mathcal{N}_{\mathcal{D}_\circ/\mathcal{D}} \longrightarrow \mathcal{N}_{\mathcal{M}_\circ/\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}_\circ}} \mathcal{O}_{\mathcal{D}_\circ}$$

is surjective.

Following the proof of Lemma 7.1, $H^1((L_{M_\circ/\mathcal{D}}^\bullet)^\vee)$ and $H^1((L_{M_\circ/\mathcal{M}}^\bullet)^\vee)$ are canonically isomorphic to the pullbacks of the normal sheaves $\mathcal{N}_{\mathcal{D}_\circ/\mathcal{D}}$ and $\mathcal{N}_{\mathcal{M}_\circ/\mathcal{M}}$, respectively, and the arrow α_1 and α_2 are induced by the identity map of $\mathcal{N}_{\mathcal{D}_\circ/\mathcal{D}}$ and (7.5), respectively. This proves the statements about the sheaves and arrows in the top horizontal line.

Parallel to the proof Lemma 7.1, we obtain that the vertical arrows are injective with locally free cokernels.

By definition of \tilde{V}_i , the first two sheaves in the lower horizontal line are $\tilde{V}|_{W_\circ}$ and $\gamma_\circ^* V_1|_{M_\circ} = \tilde{V}_1|_{W_\circ}$, and the arrow $\tilde{\beta}_2|_{W_\circ}$ is the obvious projection as stated. The statement about $\gamma_\circ^* \beta_\circ$ and β_\circ follows from [CL1, Lemm 2.8].

A direct calculation shows that $R^1 \pi_{M_\circ *} f_{M_\circ}^* \mathcal{O}(1)^{\oplus 5}$ and $R^1 \pi_{M_\circ *} f_{M_\circ}^* T_{\mathbb{P}^4}$ have rank five and four, respectively, and $\gamma_\circ^* \beta_\circ$ is surjective, therefore $\ker(\gamma_\circ^* \beta_\circ)$ is an invertible sheaf, and is isomorphic to $\ker(\alpha_2)$, using that the middle and the third vertical arrows are injective with locally free cokernel. Consequently, the cokernels of the middle and the third vertical arrows are isomorphic. These complete the proof of the Lemma. \square

We now determine the image sheaf of the third vertical arrow in (7.3). Let $\xi = [u, C] \in M_\circ$ be a closed point. By the description of M_\circ , $C = E \cup R$ such that E (resp. R) is a nodal genus one curve (resp. $R \cong \mathbb{P}^1$) and $p = E \cap R$ is a node of C , and that $u|_E = \text{const.}$ and $u|_R : R \rightarrow \mathbb{P}^4$ is a regular embedding. Let $\underline{\xi} \in \mathcal{M}$ be the image of ξ under the tautological $M_\circ \rightarrow \mathcal{M}$; let $v \in T_{\mathcal{M}, \underline{\xi}}$ be a non-zero vector normal to $\mathcal{M}_{(d)} \subset \mathcal{M}$. According to the obstruction theory, the image of $H^1(\phi_{\mathcal{X}/\mathcal{M}}^\vee)|_{\underline{\xi}}$ in $H^1(E_{\mathcal{X}/\mathcal{M}})|_{\underline{\xi}}$ is the linear span of the image of v under the composite

$$(7.6) \quad \text{ob}_{\mathbb{P}^4} : H^0((L_{\mathcal{M}}^\bullet)^\vee|_{\underline{\xi}}) \longrightarrow H^1((L_{M_\circ/\mathcal{M}}^\bullet)^\vee|_{\underline{\xi}}) \longrightarrow H^1(E_{\mathcal{X}/\mathcal{M}}|_{\underline{\xi}})$$

induced by the obstruction theory $\phi_{\mathcal{X}/\mathcal{M}}$ (it is the obstruction assignment map). Because $u|_E$ is constant, we have

$$H^1(E_{\mathcal{X}/\mathcal{M}}|_{\underline{\xi}}) = H^1(u^* T_{\mathbb{P}^4}) = H^1(\mathcal{O}_E) \otimes_{\mathbb{C}} (u^* T_{\mathbb{P}^4})|_p = H^1(\mathcal{O}_E) \otimes_{\mathbb{C}} T_{\mathbb{P}^4, u(p)}.$$

Lemma 7.3. *The linear span of the image of $v \in H^0((L_{\mathcal{M}}^\bullet)^\vee|_{\underline{\xi}})$ in $H^1(E_{\mathcal{X}/\mathcal{M}}|_{\underline{\xi}})$ is the subspace*

$$H^1(\mathcal{O}_E) \otimes_{\mathbb{C}} u_*(T_{R,p}) \subset H^1(\mathcal{O}_E) \otimes_{\mathbb{C}} T_{\mathbb{P}^4, u(p)}.$$

Proof. Let $H = u(R) \subset \mathbb{P}^4$. Since $u|_R$ is a regular embedding, $H \subset \mathbb{P}^4$ is a smooth curve. We let $u' : C \rightarrow H$ be the factorization of $u : C \rightarrow \mathbb{P}^4$. Thus $\xi' = [u', C] \in \overline{M}_1(H, d')$, where $d' = u'_*[R] \in H_2(H, \mathbb{Z})$. We let $\overline{M}_1(H, d') \rightarrow \mathcal{M}$ be the tautological projection; thus $\underline{\xi} \in \mathcal{M}$ is also the image of ξ' .

Since $H \cong \mathbb{P}^1$, it is known that there is no first order deformation of $[u', C]$ in $\overline{M}_1(H, d')$ whose image in $T_{\underline{\xi}} \mathcal{M}$ is v (cf. [Zil], [HL1]). Thus the image of v under the obstruction assignment

$$\text{ob}_H : H^0((L_{\mathcal{M}}^\bullet)^\vee|_{\underline{\xi}}) \longrightarrow H^1(E_{\overline{M}_1(H, d')/\mathcal{M}}|_{\xi'}) = H^1(\mathcal{O}_E) \otimes_{\mathbb{C}} T_{H, u'(p)}$$

is non-zero. Since $\dim H^1(\mathcal{O}_E) = \dim T_{H, u'(p)} = 1$, the linear span of $\text{ob}_H(v)$ is $H^1(\mathcal{O}_E) \otimes_{\mathbb{C}} T_{H, u'(p)}$.

Then, because the obstruction theories of moduli of stable morphisms to schemes are compatible via morphisms between schemes, we conclude that the linear span of the image $\text{ob}_{\mathbb{P}^4}(v) \subset H^1(E_{\mathcal{X}/\mathcal{M}}|_{\underline{\xi}})$ is identical to the image of the linear span of $\text{ob}_H(u')$ under the canonical

$$H^1(E_{\overline{M}_1(H, d')/\mathcal{M}}|_{\xi'}) \longrightarrow H^1(E_{\overline{M}_1(\mathbb{P}^4, d)/\mathcal{M}}|_{\xi}) = H^1(E_{\mathcal{X}/\mathcal{M}}|_{\underline{\xi}}).$$

Adding $T_{H, u'(p)} = u_*(T_{R,p})$ as subspace in $T_{\mathbb{P}^4, u(p)}$, we prove the Lemma. \square

We consider $H^1((L_{M_\circ/\mathcal{D}}^\bullet)^\vee) \rightarrow H^1(E_{\mathcal{X}/\mathcal{D}}|_{M_\circ}) = V_1|_{M_\circ}$ of the middle vertical arrow in (7.3). By Lemma 7.2, $H^1((L_{M_\circ/\mathcal{D}}^\bullet)^\vee)$ is a rank two locally free sheaf

on M_\circ , and the arrow is injective with locally free cokernel. Let $S_\circ \subset V_1|_{M_\circ}$ be associated sub-vector bundle.

We continue to denote by $\bar{\gamma} : \bar{W} \rightarrow M$ the projection (cf. Section five). We let $\bar{W}_\circ = \bar{W} \times_M M_\circ$ and $\bar{\gamma}_\circ : \bar{W}_\circ \rightarrow M_\circ$ the projection. Recall that $\bar{V}_1 = \bar{\gamma}^* V_1(-D_\infty)$. Thus $S_\circ \subset V_1|_{M_\circ}$ provides a subbundle

$$(7.7) \quad \bar{S}_\circ = \bar{\gamma}_\circ^* S_\circ(-D_\infty) \subset \bar{V}_1|_{\bar{W}_\circ}.$$

We let

$$\eta_\circ : \bar{V}_1|_{\bar{W}_\circ} \rightarrow \bar{V}|_{\bar{W}_\circ} = (\bar{V}_1 \oplus \bar{V}_2)|_{\bar{W}_\circ}$$

be the inclusion; let $j_\circ : \bar{W}_\circ \rightarrow \bar{W}$ be the open embedding, which is flat. Recall $R_{(d)} = N_{\bar{C}_{(d),b}} \bar{C}_{(d)}$ (cf. before (6.6), see also (6.8)).

Lemma 7.4. *As cycles, we have*

$$(7.8) \quad j_\circ^*[R_{(d)}] = \eta_{\circ*}[\bar{S}_\circ] \in Z_*(\bar{V}_1|_{\bar{W}_\circ}).$$

Proof. Lemma 7.1 shows that $C_{(d)} \times_W W_\circ$ is a rank two subbundle of $\tilde{V}|_{W_\circ}$. By Proposition 6.1 and Lemma 5.4, we have $\bar{C}_{(d),b} \subset \bar{V}_2$, $\bar{C}_{(d),b} \cap \bar{V}_2|_{\bar{W}_\circ} = 0_2 \times_{\bar{W}} \bar{W}_\circ$, and $\bar{C}_{(d)} \times_{\bar{W}} \bar{W}_\circ$ is a rank two subbundle of $\bar{V}|_{\bar{W}_\circ}$. Further, they fit into the commutative diagram (the left one is a Cartesian product)

$$\begin{array}{ccccc} 0_{\bar{V}} \times_{\bar{W}} \bar{W}_\circ & \longrightarrow & \bar{C}_{(d)} \times_{\bar{W}} \bar{W}_\circ & \xrightarrow{\cong} & \bar{S}_\circ \\ \downarrow \subseteq & & \downarrow \subseteq & & \downarrow \subseteq \\ \bar{V}_2|_{\bar{W}_\circ} & \xrightarrow{\bar{\beta}_1|_{\bar{W}_\circ}} & \bar{V}|_{\bar{W}_\circ} = (\bar{V}_1 \oplus \bar{V}_2)|_{\bar{W}_\circ} & \xrightarrow{\bar{\beta}_2|_{\bar{W}_\circ}} & \bar{V}_1|_{\bar{W}_\circ}, \end{array}$$

where $\bar{\beta}_1|_{\bar{W}_\circ}$ and $\bar{\beta}_2|_{\bar{W}_\circ}$ are the obvious inclusion and projection, which implies that

$$R_{(d)} \times_{\bar{W}} \bar{W}_\circ = (N_{\bar{C}_{(d),b}} \bar{C}_{(d)}) \times_{\bar{W}} \bar{W}_\circ = \eta_\circ(\bar{S}_\circ) \subset \bar{V}|_{\bar{W}_\circ}.$$

Since $j_\circ : \bar{W}_\circ \rightarrow \bar{W}$ is an open embedding,

$$j_\circ^*[R_{(d)}] = [R_{(d)} \times_{\bar{W}} \bar{W}_\circ] = [\eta_\circ(\bar{S}_\circ)] = \eta_{\circ*}[\bar{S}_\circ] \in Z_*(\bar{V}_1|_{\bar{W}_\circ}).$$

This proves the Lemma. \square

We pick a degree d regular embedding $h : \mathbb{P}^1 \rightarrow \mathbb{P}^4$, viewed as a closed point in $\tilde{\mathcal{B}}_{(d)} = \bar{M}_0(\mathbb{P}^4, d)$. We form

$$M_h = \{[u, C] \in M_\circ \mid u|_R \cong h\} \subset M_\circ.$$

Using the convention introduced in the proof of Lemma 6.4, we have that $\bar{W}_{(d)} \cup \bar{F}_{(d)} \subset \bar{V}_{2,(d)}$, where $\bar{W}_{(d)}$ is the zero-section of $\bar{V}_{2,(d)}$. We form $\bar{W}_h = \bar{W}_\circ \times_{M_\circ} M_h$ and the inclusions

$$j_h : \bar{W}_h \longrightarrow \bar{W}_{(d)} \cup \bar{F}_{(d)}, \quad \text{and} \quad J_h : \bar{V}_1|_{\bar{W}_h} \longrightarrow \bar{V}_1 \times_{\bar{W}_{(d)}} (\bar{W}_{(d)} \cup \bar{F}_{(d)}),$$

where the last term is viewed as a vector bundle over $\bar{W}_{(d)} \cup \bar{F}_{(d)}$. Since $j_h(\bar{W}_h) \cap \bar{F}_{(d)} = \emptyset$, both j_h and J_h are proper, regular embeddings; thus the Gysin map $j_h^!$ and $J_h^!$ are well-defined.

Lemma 7.5. *The constant c in (7.1) is given by the degree $c = \deg e(j_h^* \bar{V}_1 / j_h^* \bar{S}_\circ)$.*

Proof. Let $\phi_h : \overline{W}_h \rightarrow [h]$ be the projection to the point, and let $\phi_{(d)}$ be the projection that fits into the Cartesian product

$$\begin{array}{ccc} \overline{W}_h & \xrightarrow{j_h} & Z_{(d)} = \overline{W}_{(d)} \cup \overline{F}_{(d)} \\ \downarrow \phi_h & & \downarrow \phi_{(d)} \\ [h] & \xrightarrow{\iota_h} & \tilde{\mathcal{B}}_{(d)}. \end{array}$$

(Here $\phi_{(d)}$ is the composite of $Z_{(d)} \rightarrow \overline{W}_{(d)}$ mentioned after (6.7), the morphism $\tilde{\gamma}_{(d)} : \overline{W}_{(d)} \rightarrow \tilde{\mathcal{X}}_{(d)}$, and the $\rho_{(d)} : \tilde{\mathcal{X}}_{(d)} \rightarrow \tilde{\mathcal{B}}_{(d)}$ constructed in (6.9).)

Since $j_h(\overline{W}_h) \cap \overline{F}_{(d)} = \emptyset$, $j_h^! [P_{(d),2}] = 0$. Thus

$$j_h^! [P_{(d),1}] = j_h^! ([P_{(d),1}] + [P_{(d),2}]) = j_h^! 0_{\overline{V}_{1,(d)}}^! [R_{(d)}].$$

Since Gysin maps commute, we obtain

$$j_h^! [P_{(d),1}] = j_h^! 0_{\overline{V}_{1,(d)}}^! [R_{(d)}] = 0_{j_h^* \overline{V}_1}^! [j_h^! R_{(d)}] = 0_{j_h^* \overline{V}_1}^! [\overline{S}_\circ|_{\overline{W}_h}] = e(j_h^* \overline{V}_1 / j_h^* \overline{S}_\circ).$$

On the other hand, since the Gysin maps commute with proper push-forwards,

$$\phi_{h*}(e(j_h^* \overline{V}_1 / j_h^* \overline{S}_\circ)) = \phi_{h*}(j_h^! [P_{(d),1}]) = i_h^! (\phi_{(d)*} [P_{(d),1}]) = i_h^! (c[\tilde{\mathcal{B}}_{(d)}]) = c.$$

This proves the Lemma. \square

Finally, we evaluate c . Let $\mathcal{E} \rightarrow \overline{M}_{1,1}$ with $s_1 : \overline{M}_{1,1} \rightarrow \mathcal{E}$ be the universal family of (the moduli of pointed elliptic curves) $\overline{M}_{1,1}$. We form

$$\varphi : B = \overline{M}_{1,1} \times \mathbb{P}^1 \xrightarrow{\cong} M_h \subset M_\circ$$

as follows. Let q_1 and q_2 be the first and the second projections of B . We denote $q_1^* \mathcal{E} = \mathcal{E} \times_{\overline{M}_{1,1}} B$ with $q_1^* s_1 : B \rightarrow q_1^* \mathcal{E}$ be the pullback section of s_1 . Over \mathbb{P}^1 , we form $\mathcal{R} = \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ (the second projection) the constant family of \mathbb{P}^1 over \mathbb{P}^1 , and let $s_2 : \mathbb{P}^1 \rightarrow \mathcal{R}$ be its section so that the image $s_2(\mathbb{P}^1) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal. We let $q_2^* \mathcal{R} \rightarrow B$ with $q_2^* s_2 : B \rightarrow q_2^* \mathcal{R}$ the pullback family with section. We then glue $q_1^* s_1(B) \subset q_1^* \mathcal{E}$ with $q_2^* s_2(B) \subset q_2^* \mathcal{R}$ to form a B -family of genus one nodal curves, denoted by $\pi_B : \mathcal{C}_B \rightarrow B$. We define $f_B : \mathcal{C}_B \rightarrow \mathbb{P}^4$ be the composite

$$f_B : \mathcal{C}_B = q_1^* \mathcal{E} \cup q_2^* \mathcal{R} \xrightarrow{\text{ctr}} q_2^* \mathcal{R} \xrightarrow{\text{pr}} \mathbb{P}^1 \xrightarrow{h} \mathbb{P}^4,$$

where the first arrow is contracting $q_1^* \mathcal{E}$, namely it maps $q_1^* \mathcal{E}$ onto $q_2^* s_2(B) \subset q_2^* \mathcal{R}$ and is the identity on $q_2^* \mathcal{R}$; pr is the projection to the fiber of the constant family $q_2^* \mathcal{R} \rightarrow \mathbb{P}^1$, and h is the map we picked.

It is clear that $(\pi_B, f_B) : \mathcal{C}_B \rightarrow B \times \mathbb{P}^4$ is a family of stable morphisms in $\mathcal{X} = \overline{M}_1(\mathbb{P}^4, d)$, thus induces a morphism $B \rightarrow \mathcal{X}$ that factors through $M_h \subset \mathcal{X}$ and induces an isomorphism $B \cong M_h$. In the following, we will not distinguish B from M_h ; in particular, (\mathcal{C}_B, f_B) is the tautological family on $B \cong M_h \subset \mathcal{X}$.

Let $\mathcal{H}_B = \pi_{B*} \omega_{\mathcal{C}_B/B} \cong (R^1 \pi_{B*} \mathcal{O}_{\mathcal{C}_B})^\vee$ be the Hodge bundle over B . Using the family (\mathcal{C}_B, f_B) , and because of (7.3), Lemma 7.2 and Lemma 7.3, we have

$$(V_1/S_\circ)|_{M_h} = \mathcal{H}_B^\vee \otimes_{\mathcal{O}_B} q_2^* (h^* T_{\mathbb{P}^4}/T_{\mathbb{P}^1}) = \mathcal{H}_B^\vee \otimes_{\mathcal{O}_B} q_2^* N_{R/\mathbb{P}^4},$$

where $R = h(\mathbb{P}^1) \subset \mathbb{P}^4$, and N_{R/\mathbb{P}^4} is the normal bundle to R in \mathbb{P}^4 . Also, for the line bundle L on M defined before (5.14) and giving $W = \text{Total}(L)$, we have

$$L_B := L|_{M_h} = \pi_{B*} (f_B^* \mathcal{O}(-5) \otimes \omega_{\mathcal{C}_B/B}) \cong \mathcal{H}_B \otimes_{\mathcal{O}_B} (q_2^* s_2)^* f_B^* \mathcal{O}(-5).$$

Thus, $\overline{W}_h = \mathbb{P}_B(L_B \oplus \mathcal{O}_B)$, and for $j_h : \overline{W}_h \rightarrow Z_{(d)} \supset \overline{W}_{(d)}$, we have $D_B := \mathbb{P}(0 \oplus \mathcal{O}_B) = j_h^{-1}(D_\infty)$. Following the construction of \overline{V}_1 and \overline{S}_\circ , we have

$$(7.9) \quad j_h^* \overline{V}_1 / j_h^* \overline{S}_\circ \cong \bar{\gamma}_h^*(q_2^* N_{R/\mathbb{P}^4} \otimes \mathcal{H}_B^\vee)(-D_B).$$

Let $\xi \in A^1 \mathbb{P}^1$ and $\zeta \in A^1 \overline{M}_{1,1}$ be defined via $c_1(\mathcal{H}) = \frac{1}{24}\zeta$ and $c(T_{\mathbb{P}^1}) = 1 + 2\xi$, where \mathcal{H} is the Hodge bundle on $\overline{M}_{1,1}$. We calculate $c(h^* T_{\mathbb{P}^4}) = 1 + 5d\xi$, and $c(N_{R/\mathbb{P}^4}) = 1 + (5d - 2)\xi$. Let $\bar{\xi} = \bar{\gamma}_h^* q_2^* \xi$ and $\bar{\zeta} = \bar{\gamma}_h^* q_1^* \zeta \in A^1 \overline{W}_h$ be pullbacks of ξ and ζ via \overline{W}_h to \mathbb{P}^1 and to $\overline{M}_{1,1}$, respectively. We calculate

$$(7.10) \quad c(\bar{\gamma}_h^* q_2^* N_{R/\mathbb{P}^4}) = 1 + (5d - 2)\bar{\xi} \quad \text{and} \quad c(\bar{\gamma}_h^* q_1^* \mathcal{H}^\vee) = 1 - \frac{1}{24}\bar{\zeta}.$$

Let $F \in A^2 \overline{W}_h$ be the Poincare dual of the fiber class of $\bar{\gamma}_h : \overline{W}_h \rightarrow M_h$. Using $\bar{\xi}\bar{\zeta} = F$ and $\bar{\xi}^2 = \bar{\zeta}^2 = 0$, (7.10) gives

$$c(\bar{\gamma}_h^*(q_2^* N_{R/\mathbb{P}^4} \otimes q_1^* \mathcal{H}^\vee)) = 1 + ((5d - 2)\bar{\xi} - \frac{1}{8}\bar{\zeta}) - \frac{5d - 2}{12} \cdot F.$$

Hence the euler class

$$(7.11) \quad e(j_h^* \overline{V}_1 / j_h^* \overline{S}_\circ) = [-D_B]^3 + ((5d - 2)\bar{\xi} - \frac{1}{8}\bar{\zeta}) \cdot [D_B]^2 - (5d - 2)/12 \cdot F \cdot [D_B].$$

(Here we view $[D_B] \in A^1 \overline{W}_h$ as the Poincare dual of the cycle D_B in \overline{W}_h .)

Let $\tau : D_B \rightarrow B = M_h$ be the projection (isomorphism). We compute each term. First, direct calculations give $c_1[N_{D_B/\overline{W}_h}] = 5d\tau^* q_2^* \xi - \frac{1}{24}\tau^* q_1^* \zeta$, $a \cdot [D_B]^2 = \frac{-1}{24}$, and $b \cdot [D_B]^2 = 5d$. Thus the middle term in (7.11) is $\frac{1}{12} - \frac{5d}{6}$. The $c_1[N_{D_B/\overline{W}_h}]$ just calculated implies $[D_B] \cdot \tau^* q_2^* \xi = \frac{-1}{24}$ and $[D_B] \cdot \tau^* q_1^* \zeta = 5d$. Using $[D_B]^2 = c_1[N_{D_B/\overline{W}_h}]$, we obtain $-[D_B]^3 = -[D_B]^2 \cdot [D_B] = \frac{5d}{24} + \frac{5d}{24} = \frac{5d}{12}$. Finally, we calculate $-\frac{(5d-2)[F]}{12} \cdot (-[D_B]) = \frac{5d-2}{12}$. Hence (7.11) is equal to $-\frac{1}{12}$. By Lemma 7.5, $c = -\frac{1}{12}$.

Proof of Proposition 3.6. By (6.6), Lemma 6.4 and (7.2) we conclude

$$\deg 0_{\bar{\sigma}, \text{loc}}^1[\mathbf{C}_{(d)}] = 0_{\overline{V}_{2,(d)}}^1[P_{(d),1}] = c \cdot \deg c_{5d+1}(\tilde{\mathcal{K}}_{(d)})[\tilde{\mathcal{B}}_{(d)}] = \frac{(-1)^{5d}}{12} N_0(d) Q.$$

This completes the algebro-geometric proof of the hyperplane property of the reduced genus one GW-invariants of the quintics. \square

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